Sets, Mappings and Functions

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Sets

Definition

A **set** is a collection of definite, distinct objects m, concrete or imaginary, thus forming a new object M.
If m is an element of M, we write $m \in M$.

We call the set with no elements the empty set $\emptyset = \{\}$.

Any number system (the naturals, the integers, etc.) can be a set.

$\mathbb{Z}_8 = 0, 1, 2, 3, 4, 5, 6, 7$

$I = 0, 1$

If every element of the set N is also an element of the set M(that is, $\forall m \in N \rightarrow m \in M$), then N is called a subset of M (we write $N \subset M$ or $N \subseteq M$).

Two sets M, N are equal (denoted by N $=$ M), if $N \subset M$ and $M \subset N$.

Let M, N be sets. Then **the complement** of N in M is

$$M \setminus N = \{m \in M \mid m \notin N\}$$

The **union** of M and N is

$$M \cup N = \{m \in M \text{ or } m \in N\}$$

The **intersection** of M and N is

$$M \cap N = \{m \in M \text{ and } m \in N\}$$

Proposition 1

- Commutativity: $A \cup B = B \cup A$ and $A \cap B = B \cap A$
- Associativity:

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

- Distributivity:

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Proposition 2

- Identity:

$$A \cap U = A \text{ for } A \subset U$$

$$A \cup \emptyset = A$$

- Complement:

$$A \cup A' = U \text{ (A' is the complement of A)}$$

$$A \cap A' = \emptyset$$

$$A'' = A$$

Proposition 3

- Idempotence: $A \cap A = A$ and $A \cup A = A$
- Dominance: $A \cup U = U$ and $A \cap \emptyset = \emptyset$
- Absorption laws: $A \cup (A \cap B) = A$ and $A \cap (A \cup B) = A$

Using previous propositions, prove?

Cartesian Products

Definition

Given sets A and B, we can define a new set $A \times B$, called the Cartesian product of A and B, as a set of ordered pairs. That is,

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example

Let $A = \{1, 2, 3\}$, $B = \{x, y\}$, and $C = \emptyset$ then

$$A \times B = \{(1, x), (2, x), (3, x), (1, y), (2, y), (3, y)\}$$

$$A \times C = \emptyset$$

Cartesian Products

We define the Cartesian product of n sets to be

$$A_1 \times A_2 \times ... \times A_n = \{(a_1, a_2, ..., a_n) : a_i \in A_i \text{ for i} = 1,..,\text{n}\}$$

If $A_1 = A_2 = ... = A_n$, we write A^n for $A \times A.... \times A$.

Example

$\mathbb{R}^3$ consists of all of the 3-tuples of real numbers.

Cartesian Products

Subsets of $A \times B$ are called relations. A mapping $f \subset A \times B$ from a set A to a set B to be the special type of relationships such that

$$a \in A, \text{ there exists a unique } b \in B, (a, b) \in f$$

We write $f : A \to B$

We often write $f(a) = b$ instead of $(a, b) \in A \times B$. A is called **the source** and B is **the target** of f.

Mapping

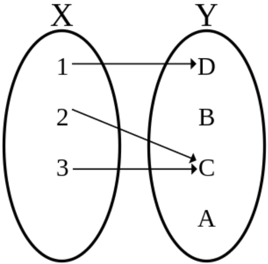
Definition

Intuitively, mapping is a process in which each element of a set X (domain) is associated with one element of a set Y (range/codomain).

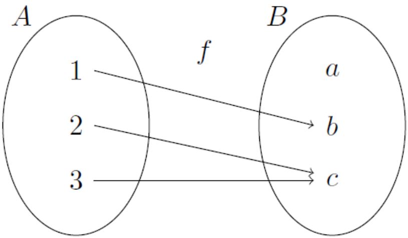
- **Domain**: the set of allowed inputs to a function.
- **Range/Codomain**: the set of possible outputs from a function.

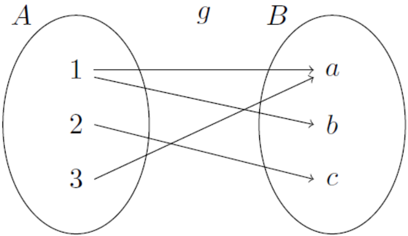
Functions

A map is often used as a synonym for a function. Only a **one-to-one** and **many-to-one** can be called a function.



Domain $X = \{1, 2, 3\}$ and codomain (range) $Y = \{A, B, C, D\}$, the function $f : X \to Y$ is defined by the set of pairs $\{(1, D), (2, C), (3, C)\}$





Which one is a function?

Which one is a function?

- $y = x^6 + 4x^3 + 1$
- $y = lnx$
- $y^3 = sinx + 1$
- $x^2 + y^2 = 8$

Functions

One-to-many and many-to-many mappings are not function.
But we still need to consider more...

Functions

For a function to exist

- the domain must be defined, or values that can not be in the domain must be identified.

Consider a function

$$y = \frac{1}{x}$$

We need to define a domain where $x \in \mathbb{R} \neq 0$. If $x = 0$, we can not associate this value with any value in the range.

Functions

Consider another function:

$$y = \sqrt{x+1}$$

- the domain must be defined, $x \geq -1 \in \mathbb{R}$

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Interval Notation

if $x \in \mathbb{R}$:

- $(a, b) =]a, b[\rightarrow a < x < b$
- $(a, b] =]a, b] \rightarrow a < x \leq b$
- $[a, b) = [a, b[\rightarrow a \leq x < b$
- $[a, b] = a \leq x \leq b$

How to find the range (but less important)?

- based on the domain values

Operations on Functions

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$(f-g)(x) = f(x) - g(x)$$

$$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$$

Suppose D_f is the domain of f and D_g is the domain of g. Then the domain of f+g, f-g, fg are the same and equal to the $D_f \cap D_g$. While the domain of f/g is $x \in D_f \cap D_g : g(x) \neq 0\}$.

Functions

How to find the inverse function $f^{-1}(x)$ of the function $f(x)$?

$$f : A \rightarrow B = \{f(x) \mid x \in A\}$$

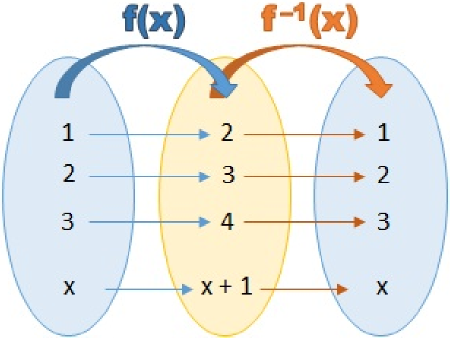
$$f^-1 : B \rightarrow A\{x \mid f(x) \in B\}$$

- Domain f(x) is equal to Range $f^{-1}(x)$
- Range f(x) is equal to Domain $f^{-1}(x)$

According to the definition, the inverse function does not always exist.

- check the mapping;
- define the domain and range;

Functions



Functions

$$f(x) = 2x + 1; f^{-1}(x) = \frac{x-1}{2}$$

$$f(x) = x^2; f^{-1}(x) = \pm\sqrt(x)$$

Do the inverse functions exist?

$$(f \circ f^{-1})(x) = (f^{-1} \circ f) = x$$

Note: Function composition $f(g(x)) = (f \circ g)(x)$
The domain of $f \circ g$ is $\{x \in D_g : g(x) \in D_f\}$, all values x in the domain of g such that g(x) is in the domain of f.

Example 1

Let $f(x) = x^2$ and $g(x) = 2x + 5$ then calculate $(f \circ g)(x)$ and $(g \circ f)(x)$. Conclude?

Function

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Let $f(x) = x^2$ and $g(x) = 2x + 5$ then calculate $(f \circ g)(x)$ and $(g \circ f)(x)$. Conclude?

Example 2

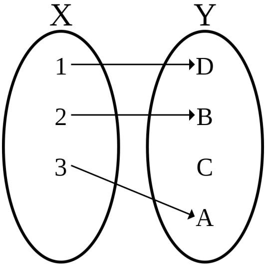
Let $f(x) = x^3 + 1$ and $g(x) = \sqrt{x} - 2$ then find the domain of $(f \circ g)(x)$ and $(g \circ f)(x)$. Conclude?

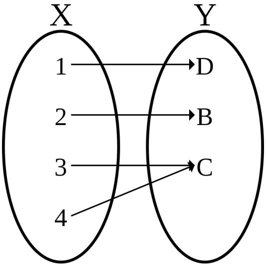
Definition

Intuitively, an injective (or one-to-one) function never maps distinct elements of its domain to the same element of its codomain.

$$f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$$

$$f(x_1) \neq f(x_2) \Leftrightarrow x_1 \neq x_2$$





Which one is an injection?

Mapping

Example

$f : X \to Y$ where f(x) = 2x +3

$$\text{Suppose } f(x) = f(y), 2x + 3 = 2y + 3, \text{ which implies } x = y$$

Other functions are injective:

$$f(x) = lnx$$

$$f(x) = \sqrt{x + 1}$$

$$f(x) = x^3 + 10x$$

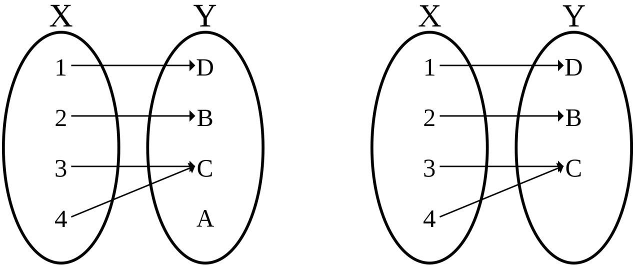
$f(x) = x^2$ is not injective, because f(1) = f(-1) = 1

Mapping

Definition

A surjective (or onto) function is defined if, for every element y in the codomain Y, there is at least one element x in the domain X such that f(x) = y.

$$\forall y \in Y, \exists x \in X, f(x) = y$$



Which one is a surjection?

Example

Surjective functions: $f : \mathbb{Z} \to \{0, 1\}$, f(n) = n mod 2
$f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1$

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 \text{ is not surjective}$$

$$f : \mathbb{R} \to \mathbb{R}^+, f(x) = x^2 \text{ is surjective}$$

Mapping

Definition

A bijective function is defined if the function is both injective and surjective.

Definition

A bijective function always has the inverse function.

Mapping

Example

Given a matrix 2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

we can define a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$T(x, y) = (ax + by, cx + dy)$$

for all $(x, y) \in \mathbb{R}^2$

A mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$ given by matrices is called linear maps or linear transformations.

Applications

For encoding and decoding:

- Encode a message using a bijective function so that the receiver can decode the encoded message;
- If there is no inverse function, the encoded message can be decoded with another meaning.

This principle is used for data conversion, transformation, projection, etc.

- Conversion between Fahrenheit and Celsius;
- Fourier Transform/Inverse Fourier Transform (analog and digital);