Review Logistic Regression

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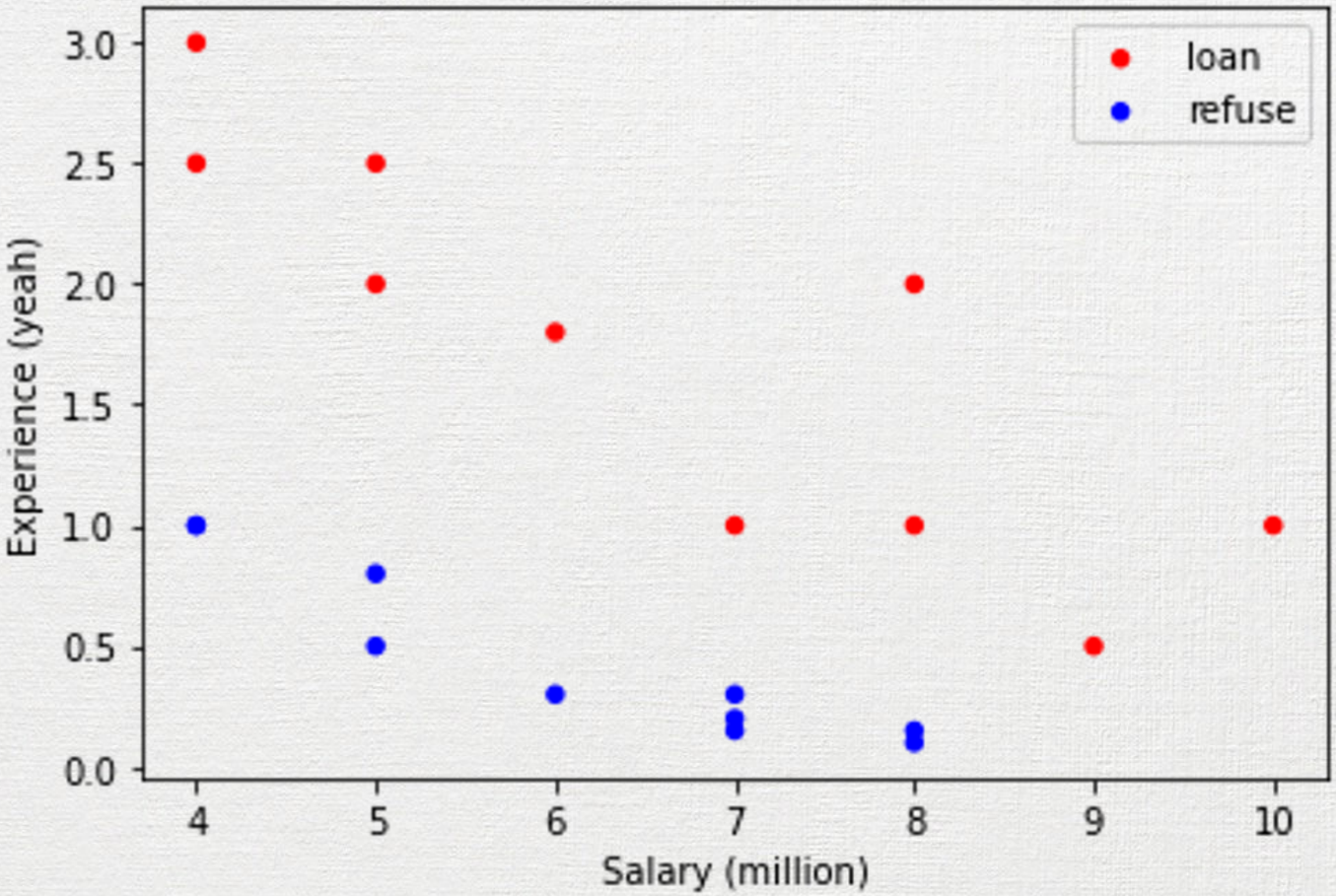
Review

Example

- Problem: decision supporting for loan program
- Input: Having data about salary and working time of employees
- Output: Loan or not?

Salary (M VND)	Experience (year)	Loan
10	1	1
9	0.5	1
5	2	1
...	...	...
8	0.1	0
6	0.3	0
7	0.15	0
...	...	...

Example



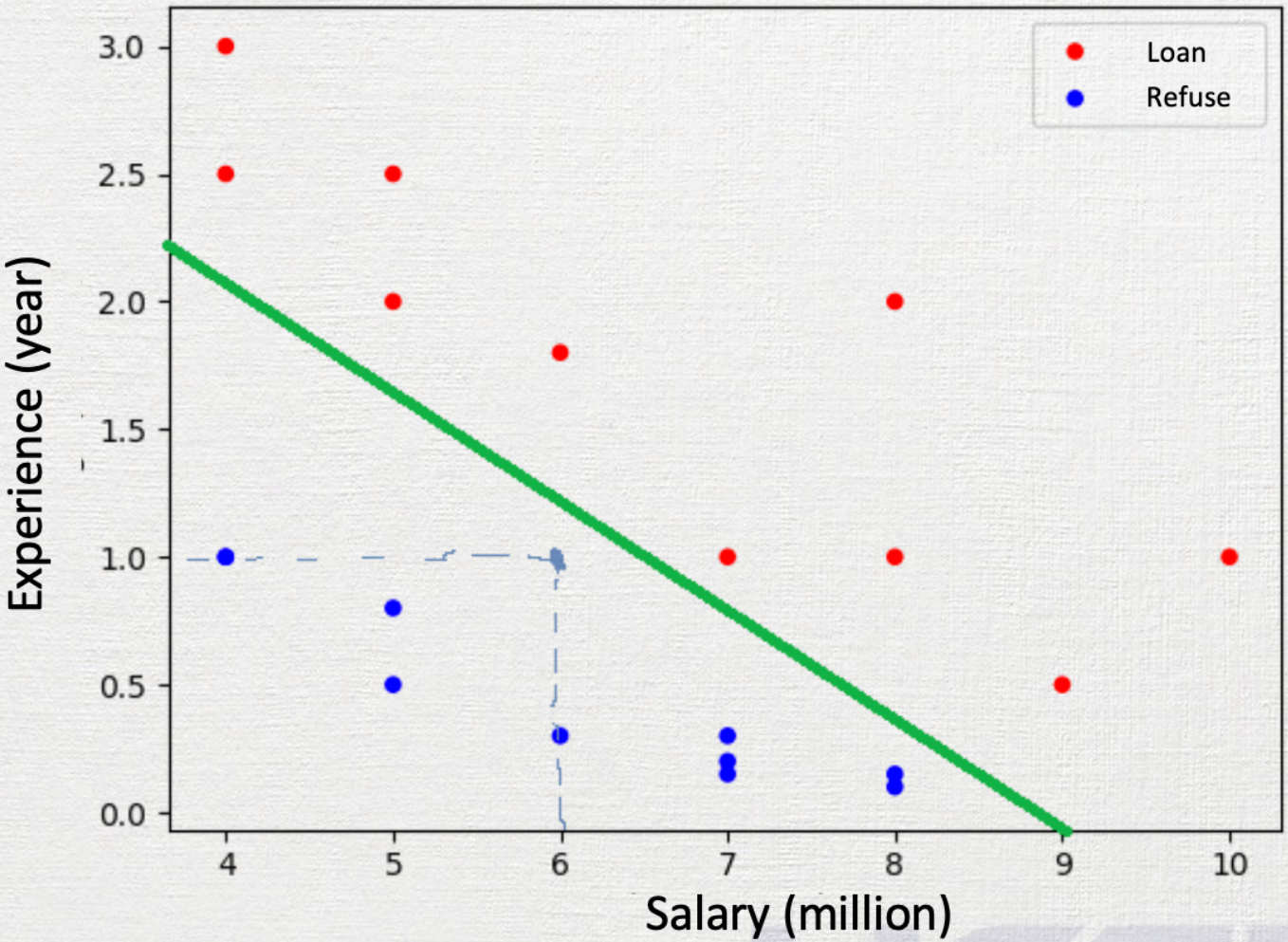
Visualization

Formulation

- Let $x_i^{(1)}$ be the salary and $x_i^{(2)}$ be the working time of the profile i
- Prediction model is defined as follows:

$$\hat{y}_i = w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0$$

Visualization



Visualization

Prediction

- Requirement: estimate the probability that a new profile should be loaned or not
- Output:
  - If the estimated loan probability >= threshold t, then the new profile should be loaned
  - Otherwise, it should be refused

Sigmoid

$$\sigma(z) = \frac{1}{1 + e^{-z}} \tag{1}$$

- Continuous function with real values in the interval $(0, 1)$
- Derivative at every point (for applying gradient descent)

Model definition

- Estimated loan probability $\hat{y}_i$ is therefore:

$$\hat{y}_i = \sigma(\hat{y}_i) = \sigma(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0) \tag{2}$$

- But $\sigma(x) = \frac{1}{1+e^{-z}}$, therefore:

$$\hat{y}_i = \frac{1}{1 + e^{-(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0)}} \tag{3}$$

Loss function

- Consider the probability that the model predicts that the profile i will be **loaned** as follows:

$$p(x^{(i)} = 1) = \hat{y}_i \tag{4}$$

- Consider the probability that the model predicts that the profile i will be **refused** as follows:

$$p(x^{(i)} = 0) = 1 - \hat{y}_i \tag{5}$$

- In total:

$$p(x^{(i)} = 1) + p(x^{(i)} = 0) = 1 \tag{6}$$

Loss function

- For each data point $(x^{(i)}, y_i)$, loss value L_i is defined as:

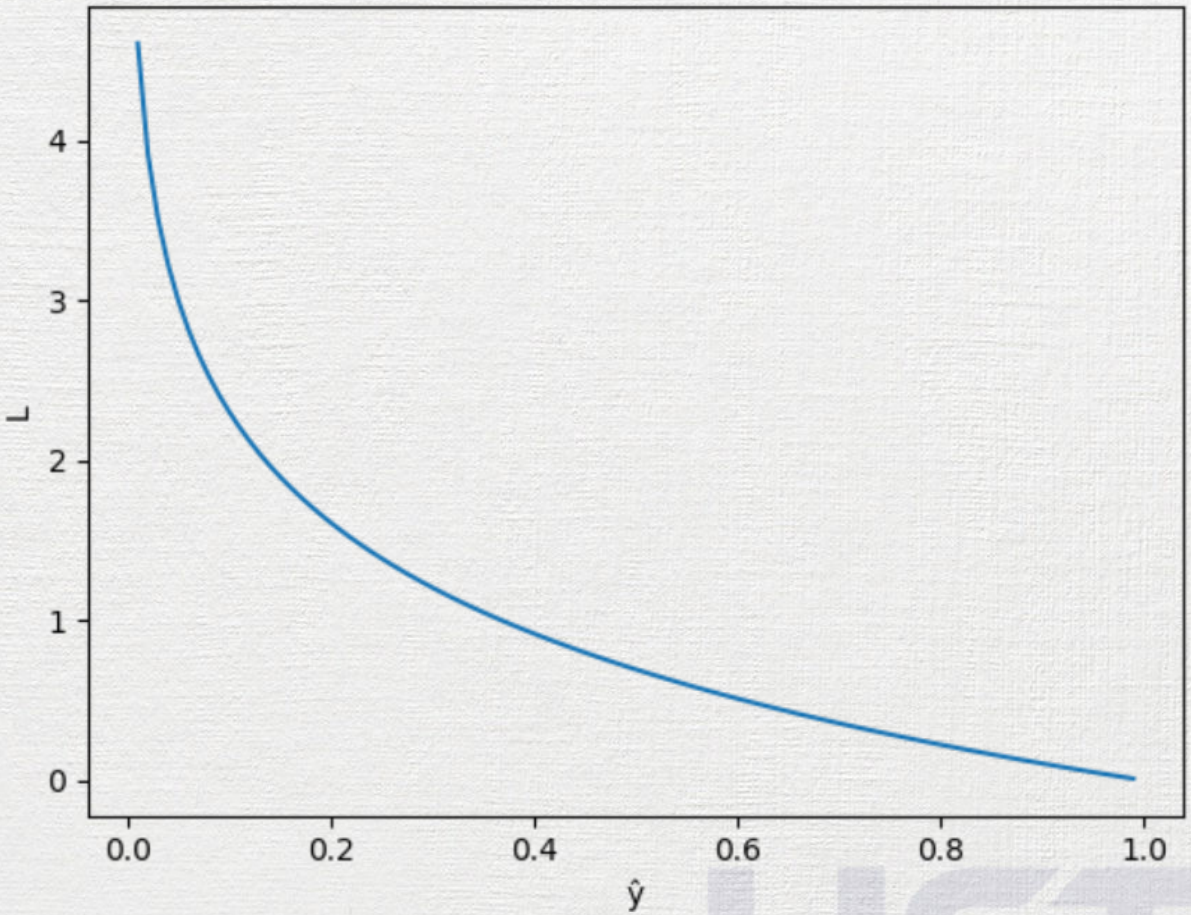
$$L_i = \begin{cases} -log(\hat{y}_i) & \text{if } y_i = 1 \\ -log(1 - \hat{y}_i) & \text{if } y_i = 0 \end{cases} \tag{7}$$

- To combine:

$$L_i = -(y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i)) \tag{8}$$

Loss function

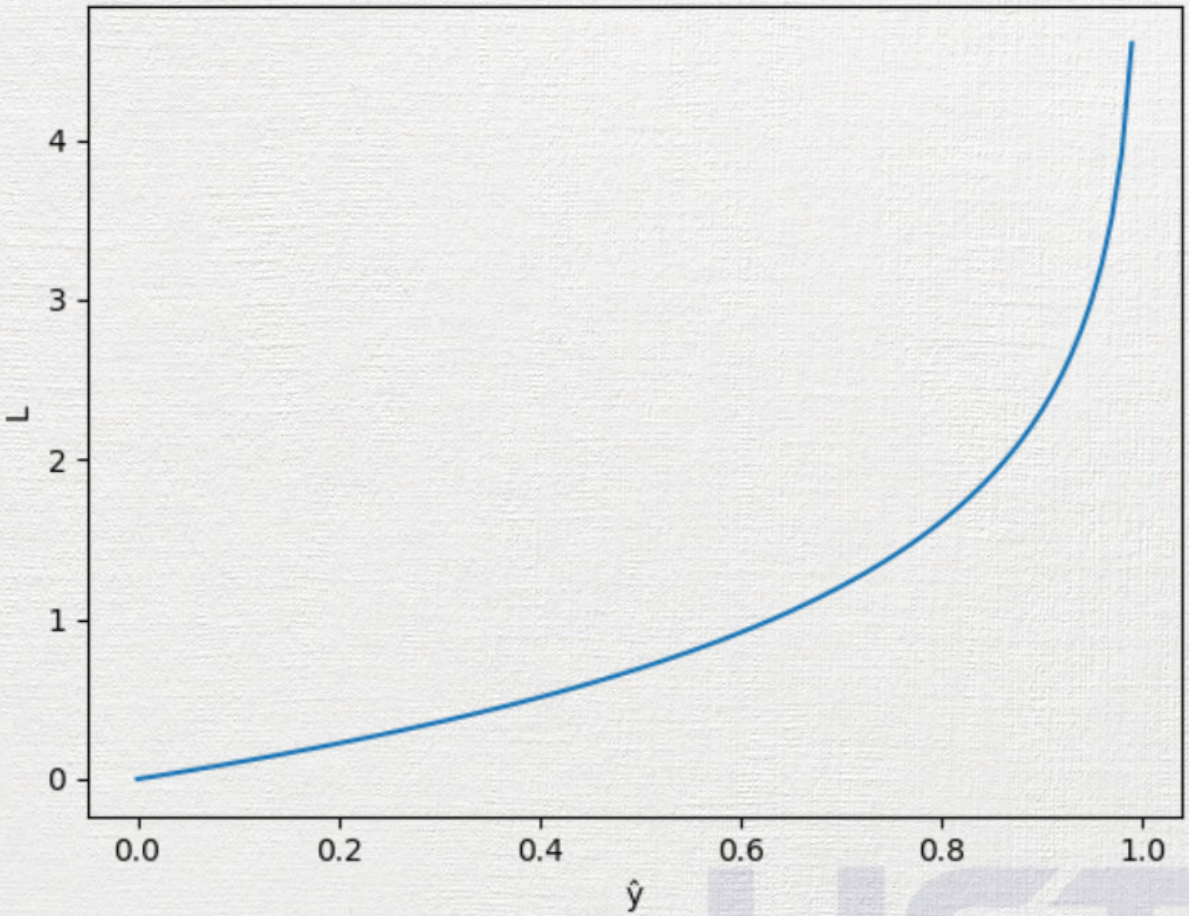
$$L_i = -(y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i))$$



$$y_i = 1$$

Loss function

$$L_i = -(y_i \log(\hat{y}_i) + (1 - y_i) \log(1 - \hat{y}_i))$$



$$y_i = 0$$

Loss function

- For all data points, loss function J is defined as:

$$J = -\frac{1}{N}\sum_{i=1}^{N}(y_i \log(\hat{y}_i) + (1 - y_i) * log(1 - \hat{y}_i)) \tag{9}$$

- Binary cross entropy loss
- Q: Why not MSE?

Training

- Function

$$J = -\frac{1}{N}\sum_{i=1}^{N}(y_i \log(\hat{y}_i) + (1 - y_i) * log(1 - \hat{y}_i)) \qquad (10)$$

- Apply gradient descent algorithm to find parameters $\{w_0, w_1, w_2\}$ which minimize J

Prediction

- Given a new profile (x_{new}, y_{new}) calculate predicted loan probability $\hat{y}_{new}$ using found parameters $\{w_0, w_1, w_2\}$
- Loan decision is defined as follows:
  - If $\hat{y}_{new} > t$, loaned
  - Else, refused

Practice!

Labwork 3: Logistic Regression

- Implement (from scratch!) logistic regression using previous gradient descent code to optimize w_0, w_1, w_2
  - Input: a CSV file with 3 columns
  - Output: w_0, w_1, w_2
  - Print the intermediate iterative steps
- Try experimenting with the loan decision example
- Write a report (in LaTeX):
  - Name it « Report.3.Logistic.Regression.tex »
  - How you implement the algorithm
  - Analyze the effect of different learning rate r w.r.t. convergence
- Push your code and report to your forked repository

Labwork 3: Logistic Regression (extras)

- Gradient descent for linear regression with binary cross entropy error loss
- Remind: single loss value with sigmoid

$$L_i = -(y_i \log \sigma(\hat{y}_i) + (1 - y_i) \log(1 - \sigma(\hat{y}_i))) \tag{11}$$

- Where
  - $\hat{y}_i$ is predicted output, $\hat{y}_i = w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0$
  - $\sigma(z)$ is sigmoid function $\sigma(z) = \frac{1}{1+e^{-z}}$

Labwork 3: Logistic Regression (extras)

$$L_i = -(y_i \log \sigma(\hat{y}_i) + (1 - y_i) \log(1 - \sigma(\hat{y}_i))) \tag{12}$$

- We have

$$y_i \log \sigma(\hat{y}_i) = y_i \log \frac{1}{1 + e^{-\hat{y}_i}} = -y_i \log(1 + e^{-\hat{y}_i}) \tag{13}$$

$$\begin{aligned}
(1 - y_i) \log(1 - \sigma(\hat{y}_i)) &= (1 - y_i) \log(1 - \frac{1}{1 + e^{-\hat{y}_i}}) \\
&= (1 - y_i) \log(\frac{e^{-\hat{y}_i}}{1 + e^{-\hat{y}_i}}) \\
&= (1 - y_i)(\log(e^{-\hat{y}_i}) - \log(1 + e^{-\hat{y}_i})) \\
&= (1 - y_i)(-\hat{y}_i - \log(1 + e^{-\hat{y}_i}))
\end{aligned} \tag{14}$$

Labwork 3: Logistic Regression (extras)

- Single loss value

$$
\begin{aligned}
L_i &= -(y_i \log \sigma(\hat{y}_i) + (1 - y_i) \log(1 - \sigma(\hat{y}_i))) \\
&= -(-y_i \log(1 + e^{-\hat{y}_i}) + (1 - y_i)(-\hat{y}_i - \log(1 + e^{-\hat{y}_i})))
\end{aligned} \tag{15}
$$

Labwork 3: Logistic Regression (extras)

$$
\begin{aligned}
L_i &= -(-y_i \log(1 + e^{-\hat{y}_i}) + (1 - y_i)(-\hat{y}_i - \log(1 + e^{-\hat{y}_i}))) \\
&= -(y_i \hat{y}_i - (\hat{y}_i + \log(1 + e^{-\hat{y}_i}))) \\
&= -(y_i \hat{y}_i - \log e^{\hat{y}_i} - \log(1 + e^{-\hat{y}_i})) \\
&= -(y_i \hat{y}_i - \log((e^{\hat{y}_i})(1 + e^{-\hat{y}_i}))) \\
&= -(y_i \hat{y}_i - \log(1 + e^{\hat{y}_i}))
\end{aligned}
\tag{16}
$$

Labwork 3: Logistic Regression (extras)

- Loss of all data points

$$J = \frac{1}{N}\sum_{i=1}^{N} L_i = -\frac{1}{N}\sum_{i=1}^{N}(y_i\hat{y}_i - \log(1+e^{\hat{y}_i}))$$

$$= -\frac{1}{N}\sum_{i=1}^{N}(y_i(w_1x_i^{(1)} + w_2x_i^{(2)} + w_0) - \log(1+e^{w_1x_i^{(1)}+w_2x_i^{(2)}+w_0})) \quad (17)$$

- Minimize this function w.r.t w_0, w_1, w_2,
  - Needs parital derivatives $\frac{\mathrm{d}J}{\mathrm{d}w_0}, \frac{\mathrm{d}J}{\mathrm{d}w_1}, \frac{\mathrm{d}J}{\mathrm{d}w_2}$

Labwork 3: Logistic Regression (extras)

- Remind: gradient descent
  - Initial value x_0
  - Function $f(x)$
  - First order derivative $f'(x)$
  - Learning rate r
  - Threshold t

Labwork 3: Logistic Regression (extras)

- 3D gradient descent for linear regression weights w_0, w_1, w_2
  - Initial value $w_0^{(0)}, w_1^{(0)}, w_2^{(0)}$
  - Function $f(w_0, w_1, w_2) = L_i$
  - First order parital derivatives $\frac{\mathrm{d}f}{\mathrm{d}w_0}, \frac{\mathrm{d}f}{\mathrm{d}w_1}, \frac{\mathrm{d}f}{\mathrm{d}w_2}$
  - Learning rate r
  - Threshold t

Labwork 3: Logistic Regression (extras)

- Function to calculate single loss value:

$$f(w_0, w_1, w_2) = -y_i(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0) + \log(1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}) \tag{18}$$

- Therefore

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}w_0} &= -y_i + \frac{e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}} \\ &= 1 - y_i - \frac{1}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}} \\ &= 1 - y_i - \sigma(-(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0)) \end{aligned} \tag{19}$$

Labwork 3: Logistic Regression (extras)

- Function to calculate single loss value:

$$f(w_0, w_1, w_2) = -y_i(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0) + \log(1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}) \tag{20}$$

- Therefore

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}w_1} &= -y_i x_i^{(1)} + \frac{x_i^{(1)} e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}} \\
&= -y_i x_i^{(1)} + x_i^{(1)}\left(1 - \frac{1}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}}\right) \\
&= -y_i x_i^{(1)} + x_i^{(1)}(1 - \sigma(-(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0)))
\end{aligned} \tag{21}$$

Labwork 3: Logistic Regression (extras)

- Function to calculate single loss value:

$$f(w_0, w_1, w_2) = -y_i(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0) + \log(1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}) \tag{22}$$

- Therefore

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}w_2} &= -y_i x_i^{(2)} + \frac{x_i^{(2)} e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}} \\
&= -y_i x_i^{(2)} + x_i^{(2)}\left(1 - \frac{1}{1 + e^{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0}}\right) \\
&= -y_i x_i^{(2)} + x_i^{(2)}(1 - \sigma(-(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0)))
\end{aligned} \tag{23}$$

Labwork 3: Logistic Regression (extras)

- Function to calculate single loss value:

$$f(w_0, w_1, w_2) = -(y_i log(w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0) + (1 - y_i) log(1 - w_1 x_i^{(1)} - w_2 x_i^{(2)} - w_0)) \tag{24}$$

- Therefore

$$\frac{\mathrm{d}L}{\mathrm{d}w_2} = -(\frac{y_i x_i^{(2)}}{w_1 x_i^{(1)} + w_2 x_i^{(2)} + w_0} + \frac{-(1 - y_i) x_i^{(2)}}{1 - w_1 x_i^{(1)} - w_2 x_i^{(2)} - w_0}) \tag{25}$$

Labwork 3: Logistic Regression (extras)

- 3D gradient descent for linear regression weights w_0, w_1
  - Step 1: Random initialization $w_0 = 0, w_1 = 1, w_2 = 2$
  - Step 2: descent...
    - $w_0 = w_0 - r * \frac{\mathrm{d}L}{\mathrm{d}w_0}$
    - $w_1 = w_1 - r * \frac{\mathrm{d}L}{\mathrm{d}w_1}$
    - $w_2 = w_2 - r * \frac{\mathrm{d}L}{\mathrm{d}w_2}$
  - Step 3: compute $f(w_0, w_1, w_2)$. Still big?
    - Move to point (w_0, w_1, w_2)
    - Repeat Step 2