# LECTURE NOTES
# ELECTRIC CIRCUITS I
## (Basic Electrical Engineering)

### COURSE INFORMATION

**Academic field:** ENERGY

**Level:** Bachelor 2

**ECTS:** 04

**Semester/Year:** 1/ 2016-2017

### TEXT BOOK AND REFERENCE

[1] J. David Irwin, R. Mark Nelms, **Basic Engineering Circuit Analysis**, 2008 John Wiley & Sons Inc.
[2] John O'Malley, **Schaum's Outline of Theory and Problems of Basic Circuit Analysis**, Second edition, McGraw-Hill
[3] Charles K. Alexander, Matthew N. O. Sadiku, **Fundamentals of Electric Circuits**, fifth edition, McGraw-Hill, ISBN 978-0-07-338057-5.

### INSTRUCTOR

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NGUYEN Xuan Truong (1983) was born in Thanh Hoa, Vietnam. He received his Engineer diploma degree in electrical engineering from the Hanoi University of Science and Technology, Vietnam in 2007; M.S degree in electrical engineering from the Grenoble Institute of Technology, Grenoble, France, in 2009 and Ph.D. degree in electrical engineering from the Paul Sabatier University, Toulouse, France, in 01/2014. He is currently lecturer and researcher of the Energy department, at the University of Science and Technology of Hanoi (USTH), Vietnam. His research interests are in the fields: distributed generation (particularly photovoltaics), building energy management, energy conservation and efficiency, micro smart-grid; electric vehicle charging station, electric vehicle-grid integration and its services. He works also in the Clean Energy and Sustainable Development Laboratory, co-founded the electrical engineering research team.

# CONTENT

# THEORETICAL COURSE CONTENTS

| | |
|---|---|
| 1 | **Fundamental Circuits Quantities**, (Charge, Current, Voltage, Energy, Power, Sources; Circuit elements (active and passive) |
| 2 | **Basic Law**: Ohm's Law, Kirchhoff's Current Law (KCL), & Kirchhoff's Voltage Law (KVL), Series & Parallel Resistance and Voltage & Current Division |
| 3 | **DC Circuit Analysis**: Nodal Analysis (Node voltage method), Loop Analysis (mesh current method) |
| 4 | **Operational Amplifier**: Op-Amp Operation, Popular Op-Amp Circuit, Circuit with Multiple Operational Amplifiers |
| 5 | **Network Theorems**: Superposition, Thevenin and Norton equivalent circuits, Maximum Power Transfer Theorem |
| 6 | **First-order circuits** (RC, RL circuits) |
| 7 | **Second-order circuits** (RLC circuits) |

# EXPERIMENTAL COURSE CONTENTS

Basic Measurement Techniques. Series Circuits and KVL. Parallel Circuits and KCL. Loading Effect and Whetstone Bridge, R/2R Ladder Network. Internal Resistance, Thevenin Theorem and Norton Theorem. Power Transfer Theorem. Voltage/Current behavior in Capacitors. Voltage Current behaviors in Inductors.

# GRADING SYSTEM (…/20)

| | | |
|---|---|---|
| 1 | Attendance/Attitude | 10 % |
| 2 | Exercise(s) | 10 % |
| 3 | Practical (+ report) | 15 % |
| 4 | Homework (+ report) | 25 % |
| 5 | Final exam | 40 % |

# IMPORTANAT RULES

You must attend **AT LEAST 90%** of the Practical Part (Lab.) **TO PASS** the course. If you missed the attendance of an experiment, the result of its Report, Quiz and Other Works would be **ZERO.** The failure in the Practical and Assignment Parts of the Course results in an **F (<10/20)** in the **OVERALL GRADE**.

# Introduction

The lecture notes are written for internal use only (USTH – lecturer and student).

The following lecture notes are based on the topics described in the official syllabus of ELECTRICAL CIRCUITS I. The target audience for these lecture notes is students in the B2 in Energy major of the Undergraduate school in USTH.

These lecture notes contain five main chapters. They cover most of the major topic areas, but are far from comprehensive in scope. Students are still encouraged to consult one of the recommended text books of [1], [2], and [3] for in depth explanations.

These lecture notes are devoted to DC circuits. The first two chapters, it covers the passive and active elements, fundamental laws (Ohm, Kirchhoff) and theorems, circuit analysis techniques. An introduction to operational amplifiers is presented in Chapter 3. A few examples and homework problems in later chapters must be skipped. In the last chapter we perform what is normally referred to as a transient analysis. We begin the analysis with first-order circuits—that is, those that contain only a single storage element. When only a single storage element is present in the network, the network can be described by a first-order differential equation. The analysis of first-order circuits begins with the presentation of two techniques for performing a transient analysis: the differential equation approach, in which a differential equation is written and solved for each network, and a step-by-step approach, which takes advantage of the known form of the solution in every case. In the second-order case, both an inductor and a capacitor are present simultaneously, and the network is described by a second-order differential equation. Although the RLC circuits are more complicated than the first-order single storage circuits, we will follow a development like that used in the first-order case.

These lecture notes are a labor of Dr. Nguyen Xuan Truong, lecturer at Energy department of University of Science and Technology of Hanoi. Many people have contributed, and I would like to thank Dr. Tran Thanh Son from Electric Power University, Dr. Nguyen Dinh Quang from Institute Energy Science/VAST (Co-Director, Energy department – USTH). This is a first version and I hope these lecture notes will be improved with the contributions of additional instructors in the 2nd version.

*Hanoi, October 2016 – revised, October 2017.*

Instructor

**Dr. Nguyen Xuan Truong**

# CHAPTER 1: Fundamental Circuits Quantities and Basic Laws

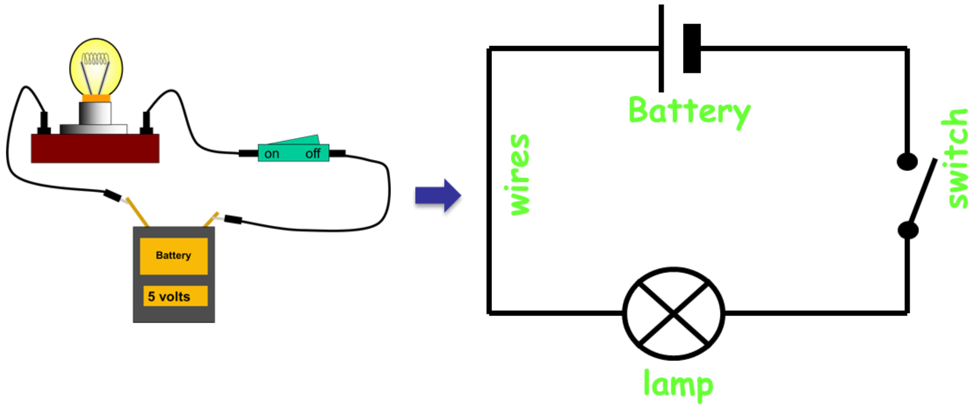
## 1. Electric Circuit

In electrical engineering, we are often interested in communicating or transferring energy from one point to another. To do this requires an **interconnection** of **electrical devices**. Such interconnection is referred to as an **electric circuit**, and each component of the circuit is known as an **element**.

Definition: An electric circuit is an interconnection of electrical elements.

**Basic quantities**: current, voltage, power and energy.

Scientists usually draw electric circuits using symbols:




For example: an electric circuit with battery, lamp, and wire-conductor.

## 2. Circuit Variables

### a. Electric Charge

The most elementary quantity in an electric circuit is the **electric charge.** The concept of electric charge is the underlying principle for all electrical phenomena. Charge is an electrical property of the atomic particles of which matter consists $q(t)$, measured in coulombs [C]. The charge of an electron is: $q = -1.602 \times 10^{-19}$ [C].

Two types of electric charge: **Positive charge** is carried by subatomic particles called protons, and **Negative charge** by subatomic particles called electrons.

**Law of Conservation of Charge**: Charge can neither be created nor destroyed, only transferred.

### b. Electric Current

Electric current in a wire $i(t)$ is defined as the net amount of charge that passes through the wire per unit time, (other: the time rate of change of charge), and is measured in amperes (A).

$$i(t) = \frac{dq(t)}{dt}$$

**<u>Note 1:</u>**

- 1 ampere (A) = 1 coulomb/second (C/s).
- The charge transferred between time $t_1$ and $t_2$ is obtained by:

$$q(t) = \int_{t1}^{t2} idt$$

- To talk about current, we need to specify two things: **direction; amount**

  They are conveyed by: **arrow; value (positive/negative)**

- Current in circuits physically realized by movement of electrons.

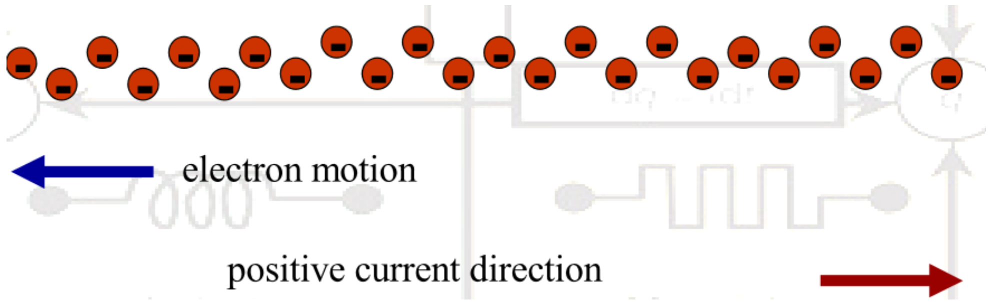
- Direction of current must be specified by an arrow.

- By convention, <u>current direction defined as flow of positive charge</u>.

- *Note that positive charge is not flowing physically.*

- Electrons have negative charge.

- They move in the opposite direction of current.

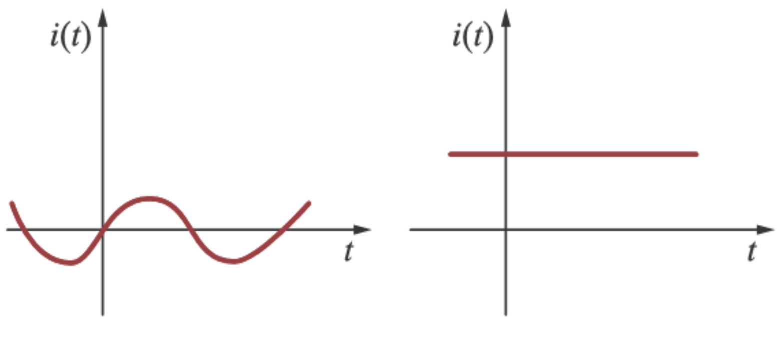



**Two kinds of Electric Current:**

- An **alternating current (AC)** is a current that varies with time --- Figure a.

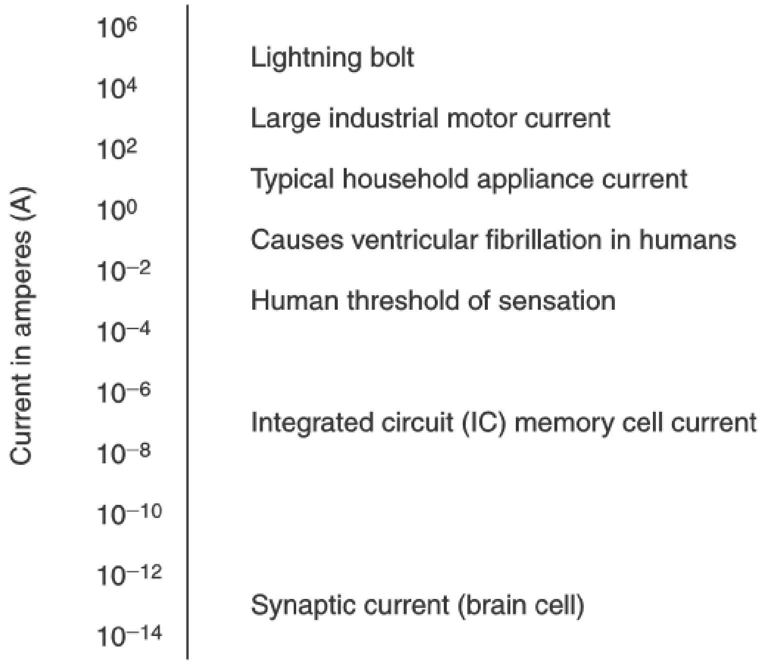
Such AC current is used in your household, to run the air conditioner, refrigerator, washing machine, and other electric appliances.

- A **direct current (DC)** is a current that remains constant with time --- Figure b.




(a) (b)

## Note 2: Typical current magnitude

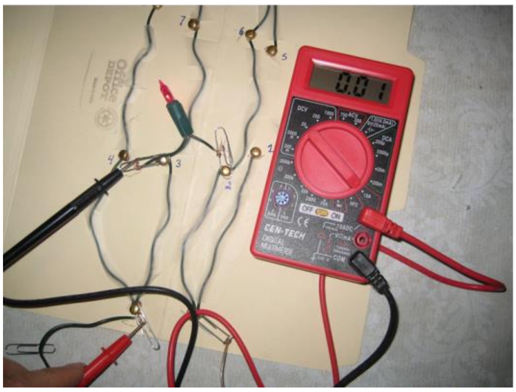


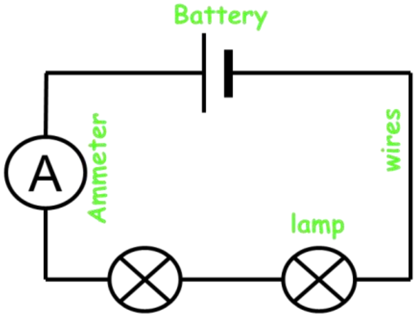

## Note 3: Current range and Effect on person

| $I_{ng}$,[mA] | **Tác hại đối với người** | |
|---|---|---|
| | **AC current, *f* = (50 - 60)[Hz]** | **DC current** |
| 0,6 - 1,5 | Faint tingle *(Bắt đầu thấy tê)* | Generally, not perceptible (*Chưa có cảm giác)* |
| 2 - 3 | Increase stinging sensation (*Tê tăng mạnh)* | Generally, no sensation (*Chưa có cảm giác)* |
| 5 - 7 | Muscle starts clamping (*Bắp thịt bắt đầu co)* | Stinging sensation *(Đau như bị kim đâm)* |
| 8 - 10 | Hand sticks to electrical object (*Tay không rời vật có điện)* | Increase temperature *(Nóng tăng dần)* |
| 20 - 25 | Hand sticks to electrical object, getting hard to breath *(Tay không rời vật có điện, bắt đầu khó thở)* | Muscle lamps and vibrates *(Bắp thịt co và rung)* |
| 50 - 80 | Respiratory arrest, increase heart beat *Tê liệt hô hấp, tim bắt đầu đập mạnh* | Difficult to breath and to take the hand off the object *(Tay khó rời vật có điện, khó thở)* |
| 90 - 100 | More than 3s, heart stops beating *(Nếu kéo dài với t ≥ 3[s] tim ngừng đập)* | Respiratory arrest *(Hô hấp tê liệt)* |

**Note 4: Measuring current**

Electric current is measured in amps (A) using an **ammeter** connected in series in the circuit






### c. Voltage (or potential difference)

Voltage is the energy absorbed or expended as a unit charge moves from one point to the other.

- Sometimes called potential difference.

- Can be created by a separation of charge.

- Is a measure of the potential between two points.

- Voltage pushes charge in one direction.

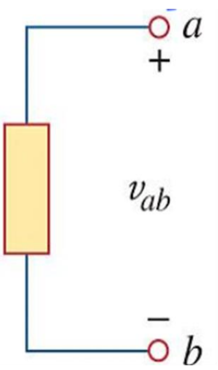
- We use polarity (+ and – on batteries) to indicates which direction the charge is being pushed

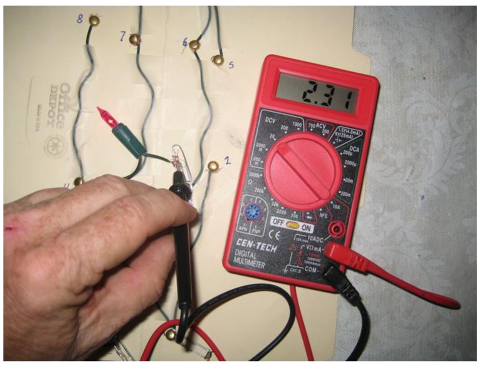
Voltage is the energy required to move a unit charge through an element, measured in volts (V). The voltage between two points a and b in a circuit is denoted by $v_{ab}$ can be interpreted in two ways:

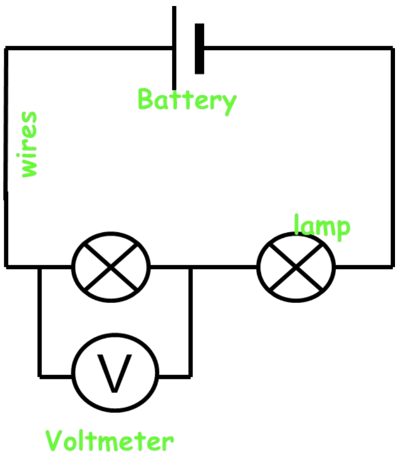
+ point a is at a potential of $v_{ab}$ volts higher than point b,

+ or the potential at point a with respect to point b is $v_{ab}$ .

$v_{ab} = \frac{dw}{dq}$; $w$ is the energy in joules [J] and $q$ is charge in coulombs [C]



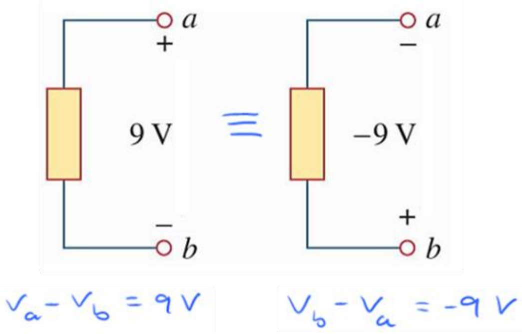


The 'electrical push' which the cell (battery) gives to the current is called the voltage. It is measured in volts (V) on a **<u>voltmeter</u>**

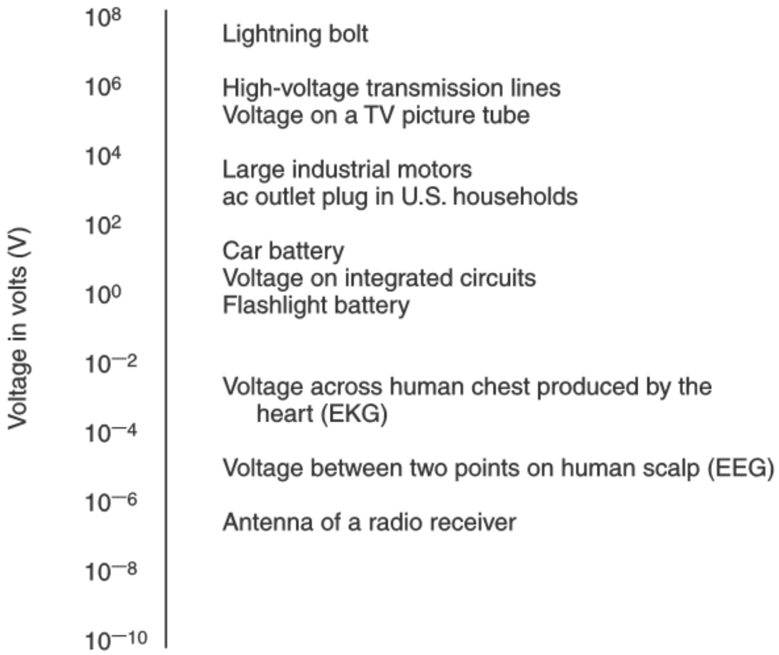
**To talk about voltage, we need to specify:** polarity (+, -); and value (positive or negative number)

The (+) and (-) signs at the point a and b are used to define reference direction or voltage polarity.

<u>For example</u>:




**<u>Note: Typical voltage magnitude</u>**




## d. Electrical Power

Is: time rate of expending or absorbing energy, measured in watts (W). Mathematically, the instantaneous power:

$p = \frac{dw}{dt} = \left(\frac{dw}{dq}\right) * \left(\frac{dq}{dt}\right) = v * i$; Where $p$ power in watts; $w$ = energy in Joules; $t$ = time in seconds.

**<u>Note 1: Sign of power</u>**

• **Plus sign**: Power is absorbed by the element (resistor, inductor)

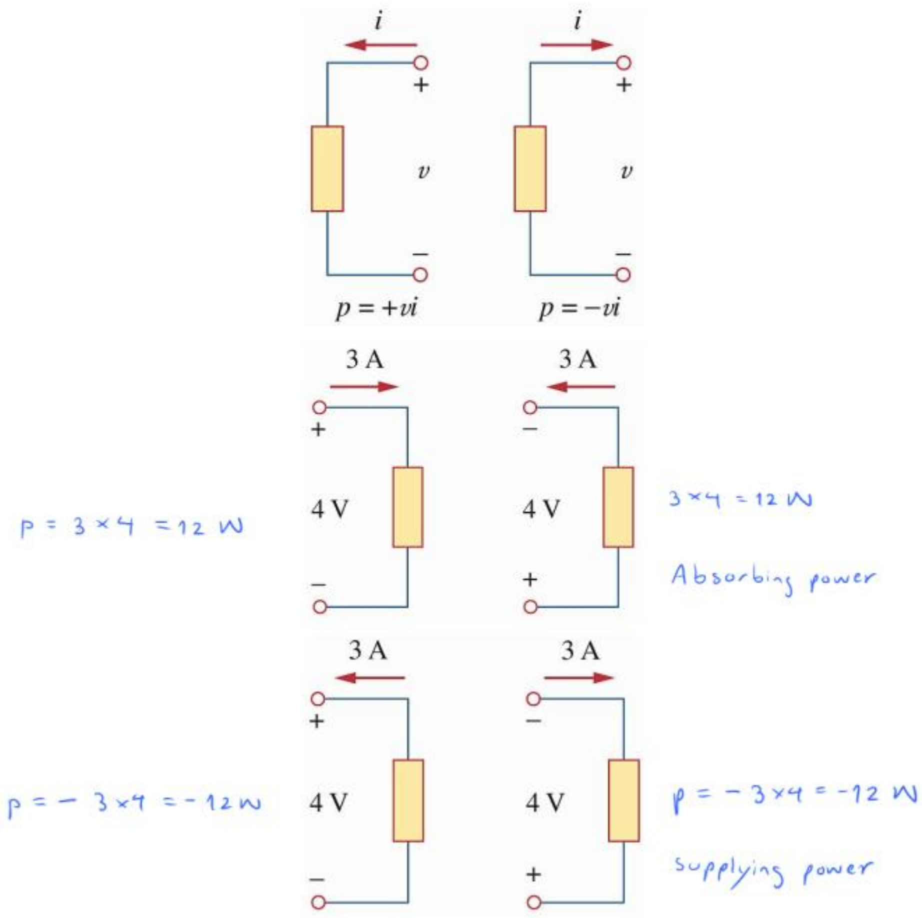
• **Minus sign**: Power is supplied by the element (battery, generator)

**Passive sign convention:**

• *If the current enters through the positive polarity of the voltage:* $p = vi$

• *If the current enters through the negative polarity of the voltage:* $p = -vi$

For example:




**Note 2: Law of Conservation of Energy:**

Energy can neither be created nor destroyed, only transferred. For this reason, the algebraic sum of power in a circuit, at any instant of time, must be zero. The total power supplied to the circuit must balance the total power absorbed.

**e. Energy**

The energy absorbed or supplied by an element from time 0 to $t$ is:

$w = \int_0^t p\,dt = \int_0^t v * i\,dt$ ----- Integration suggests finding area under the curve. Need to be careful with negative area.

*The electric power utility companies measure energy in kilowatt-hours [kWh].*

**3. Circuit Elements**

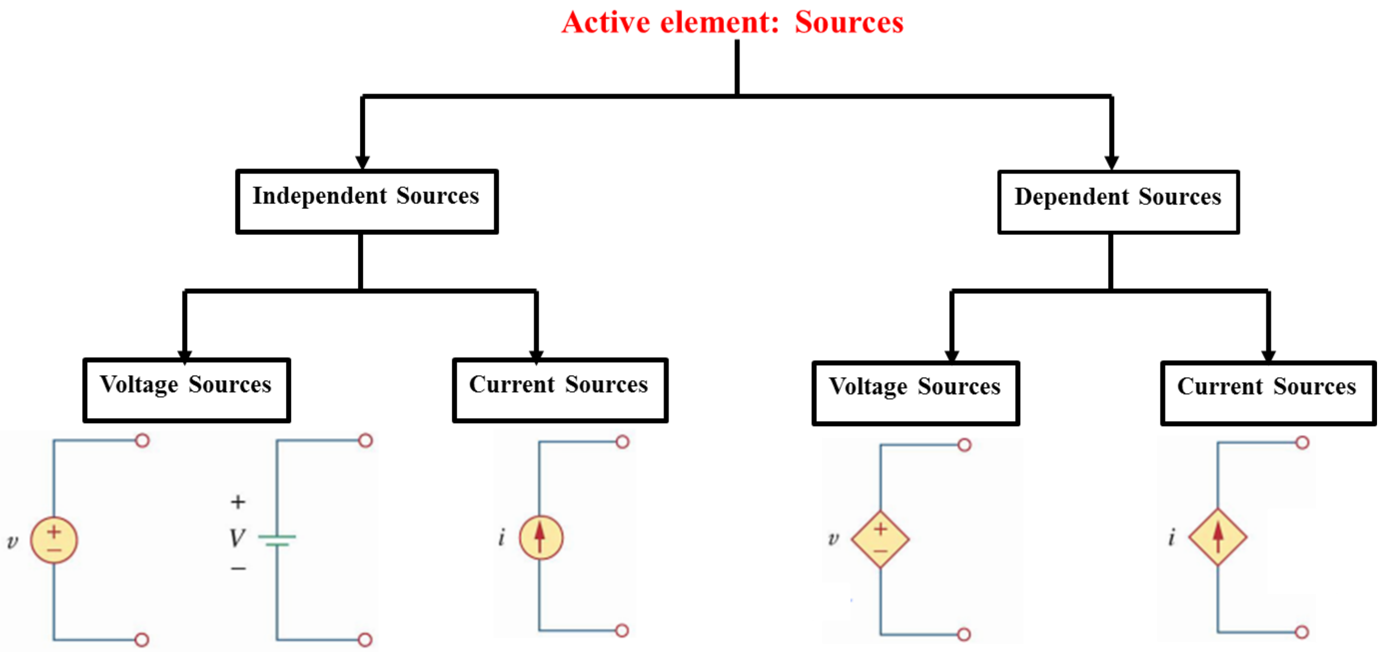
There are 2 types of elements found in electrical circuits.

+ **Active elements**: is capable of generating energy (generators, batteries, and operational amplifiers).

+ **Passive element**: cannot generate energy (resistors, capacitors and inductors).

### a. Active Elements

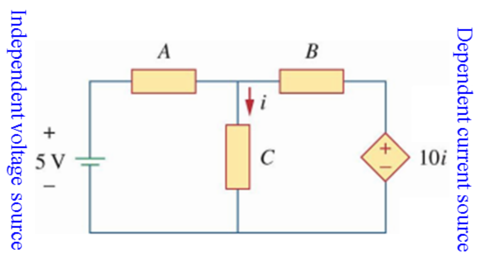
The most important active elements are **voltage and current sources**: (a) Voltage source provides the circuit with a specified voltage; (b) Current source provides the circuit with a specified current.




**Independent source:** An active element that provides a specified voltage or current that is completely independent of other circuit elements.

**Dependent source:** An active element in which the source quantity is controlled by another voltage or current.

For example:




There are four possible types of dependent sources, namely:

* A voltage-controlled voltage source.

* A current-controlled voltage source.

* A voltage-controlled current source.

* A current-controlled current source.

**b. Passive Elements**

**1) Resistors**

The ability of a material to resist the flow charge is called its resistivity. It is represented by the letter R.

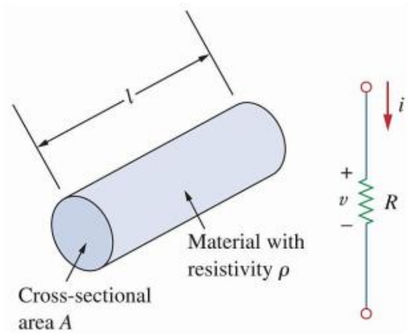
A resistor is a circuit element that dissipates electrical energy (usually as heat)

Real-world devices that are modeled by resistors: incandescent light bulbs, heating elements, long wires.

Resistance is measured in Ohms (Ω)

Resistor is indicated by the symbol:  or 

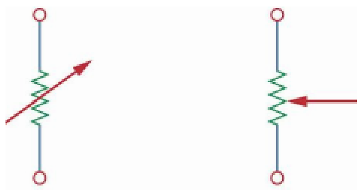
Resistance of a wire depends on some factors like as length L [m], cross- sectional area A [$m^2$] and resistivity of material ρ [Ω.m]: $R = \rho \frac{L}{A}$ [Ω]




The conductance [G] of a pure resistor is the reciprocal of its resistance. The unit of conductance is the Siemens [S]: $G = \frac{1}{R}$

A resistor is either fixed or variable. Most resistors are of the fixed type, meaning their resistance remains constant.

A common variable resistor is known as a potentiometer or pot for short.



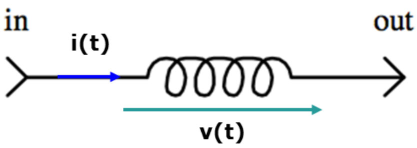
Note 1: Resistor Color Code: http://www.allaboutcircuits.com/tools/resistor-color-code-calculator/

## 2) Inductors

The circuit element that stores energy in a magnetic field is an inductor (also called an inductance). With time-variable current, the energy is generally stored during some parts of the cycle and then returned to the source during others.



**Inductance L [H]** results from the magnetic field around a current-carrying conductor; the electric current through the conductor creates a magnetic flux. Inductance is determined by how much magnetic flux $\Phi$ through the circuit is created by a given current: $L = \frac{d\Phi}{di}$




**Voltage:**

$$v_L(t) = \frac{d\Phi(t)}{dt} = \frac{d\Phi}{di} * \frac{di}{dt}$$

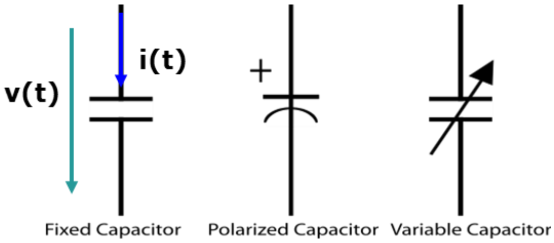
$$= L * \frac{di}{dt}$$

**Power:** $p(t) = v(t) * i(t) = L * \frac{di}{dt} * i(t)$

**Stored energy:** $W_L = \int_{-\infty}^{t} p(t)dt = \frac{1}{2} L * i^2(t)$, is equal to the amount of work required to establish the current through the inductor, and therefore the magnetic field.

## 3) Capacitors

The circuit element that stores energy in an electric field is a capacitor (also called capacitance). When the voltage is variable over a cycle, energy will be stored during one part of the cycle and returned in the next. While an inductance cannot retain energy after removal of the source because the magnetic field collapses, the capacitor retains the charge and the electric field can remain after the source is removed. This charged condition can remain until a discharge path is provided, at which time the energy is released. The charge, $q = C * v(t)$, on a capacitor results in an electric field in the dielectric which is the mechanism of the energy storage. In the simple parallel-plate capacitor there is an excess of charge on one plate and a deficiency on the other. It is the equalization of these charges that takes place when the capacitor is discharged. The power and energy relationships for the capacitance are as follows.


$$i_C(t) = \frac{dq(t)}{dt} = \frac{dq}{dv} * \frac{dv}{dt} = C * \frac{dv}{dt} \quad ; \; C = \frac{dq}{dv}$$

$$v_C(t) = \frac{1}{C}\int_{-\infty}^{t} i dt$$

$$p(t) = C\frac{dv(t)}{dt} * v(t)$$

$$W_C = \int_{-\infty}^{t} p(t)dt = \frac{1}{2}C * v^2(t)$$

*Note: Inductors, Capacitors → more detail in next part, EC 2.*

### c. Measuring Devices

**Ohmmeter:** measures the resistance of the element. Important rule: Measure the resistance only when the element is disconnected from circuits.

**Ammeter:** connected in series with an element to measure current flowing through that element. Since an ideal ammeter should not restrict the flow of current, (i.e., cause a voltage drop), an ideal ammeter has zero internal resistance.

**Voltmeter:** connected in parallel with an element to measure voltage across that element. Since an ideal voltmeter should not draw current away from the element, an ideal voltmeter has infinite internal resistance.

## 4. Basic Laws

Here we explore two fundamental laws that govern electric circuits (Ohm's law and Kirchhoff's laws) and discuss some techniques commonly applied in circuit design and analysis.

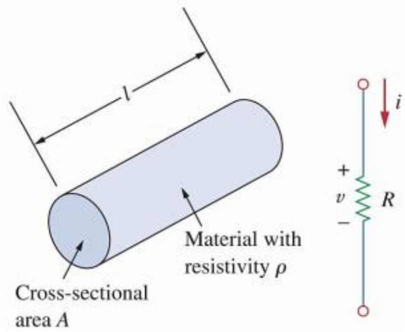
*Ohm's law coupled with Kirchhoff's two laws gives a sufficient, powerful set of tools for analyzing a large variety of electric circuits.*

### 4.1. Ohm's Law

Ohm's law shows a relationship between voltage and current of a **resistive element** such as conducting wire or light bulb. The voltage $v(t)$ across a resistor is directly proportional to the current $i(t)$ flowing through the resistor.

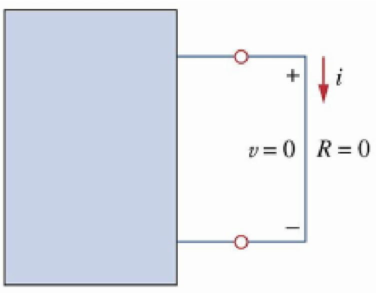
$v = i * R$; Where R = resistance of the resistor, denoting its ability to resist the flow of electric current. The resistance is measured in ohms (Ω).

**Note 1:** To apply Ohm's law, the direction of current $i$ and the polarity of voltage $v$ must conform to the passive sign convention. This implies that current flows from a *higher potential to a lower potential in order for* $v = i * R$. If *current flows from a lower potential to a higher potential,* $v = -i * R$.



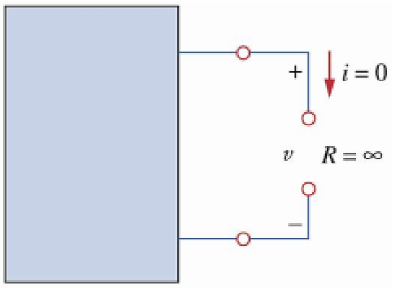

**Note 2: The two extreme possible values of R.**

(i) When R = 0, we have a short circuit and $v = i * R = 0$ showing that $v = 0$ for any $i$.



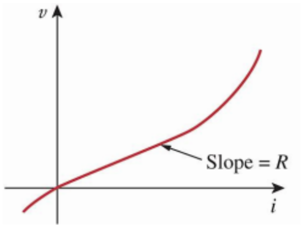
(ii) When R = ∞, we have an <u>open circuit</u> and:

$i = \lim_{R\to\infty} \frac{v}{R} = 0$; indicating that $i = 0$ for any $v$.




*Not all resistors obey Ohms law. A resistor that obeys Ohms law is known as a linear resistor*

+ A nonlinear resistor does not obey Ohms law:




Examples of devices with nonlinear resistance are the lightbulb and the diode. Although all practical resistors may exhibit nonlinear behavior under certain conditions, we will assume in this class that all elements actually designated as resistors are linear.

**Note 3: Current through a resistor causes power dissipation**

$$P = I * V = \frac{V^2}{R} = I^2 * R$$

The **power rating** *is the maximum allowable power dissipation in the resistor. Exceeding this power rating leads to overheating and can cause the resistor to burn up*.

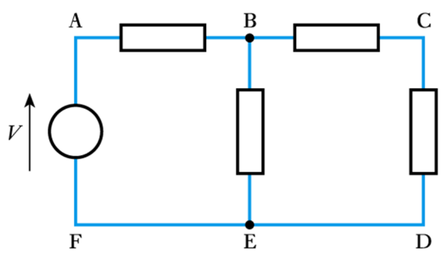
### 4.2. Kirchhoff's Laws

#### a) Concept: Node, Branch and Loop

Since the elements of an electric circuit can be interconnected in several ways, we need to understand some basic concept of network topology.

• *Network = interconnection of elements or devices*

• *Circuit = a network with closed paths*



**Branch:** A branch (b) represents a single element such as a voltage source or a resistor. A branch represents any two-terminal element. On the Figure: branches AF, AB, BE, BC, CD

**Node:** A node (n) is the point of connection between two or more branches. It is usually indicated by a dot in a circuit. On the Figure: nodes A, B, C and E

If a short circuit (a connecting wire) connects two nodes, the two nodes constitute a single node.

**Loop:** A loop (l) is any closed path in a circuit. A closed path is formed by starting at a node, passing through a set of nodes and return. On the Figure: the paths ABEFA, BCDEB and ABCDEFA are loops

**Mesh:** a loop that contains no other loop → *a loop is said to be independent*. Independent loops or paths result in independent sets of equations. A network with b branches, n nodes, and l or m independent loops will satisfy the fundamental theorem of network topology:

b = l + n −−1.

### b) Kirchhoff's Current law (KCL)

The algebraic sum of current entering a node (or a closed boundary) is zero: $\sum_{n=1}^{N} i = 0$

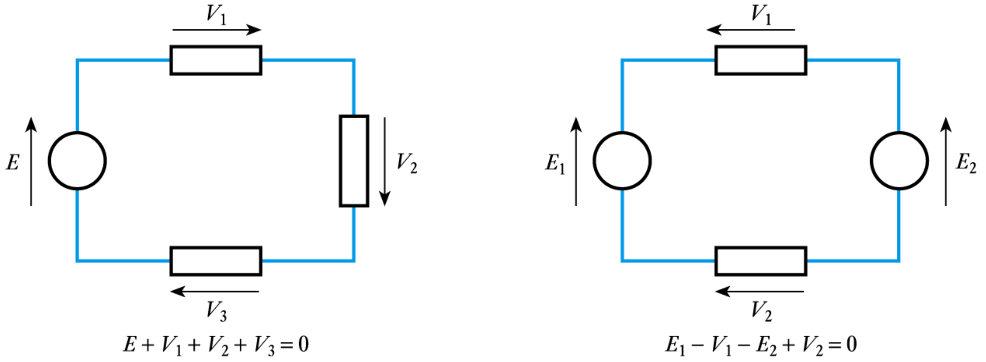
KCL is based on the **law of conservation of charge**. An alternative form of KCL is *"Sum of currents (or charges) entering a node = Sum of the currents (charges) leaving the node"*.

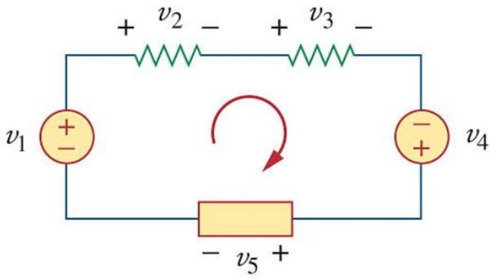

$I_1 + I_2 + I_3 + I_4 = 0$ $\qquad$ $I_1 - I_2 + I_3 - I_4 + I_5 = 0$

### c) Kirchhoff's Voltage law (KVL)

The algebraic sum of all voltages around a closed path (or loop) is zero: $\sum_{n=1}^{N} v = 0$

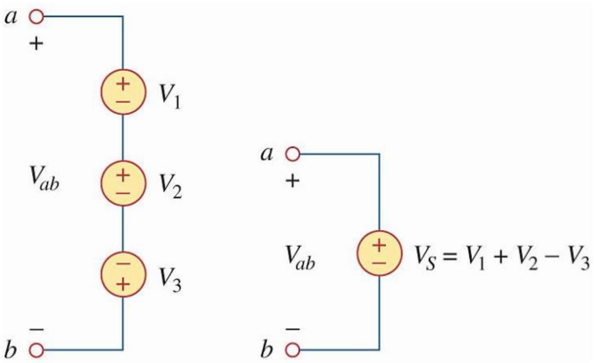
KVL is based on the **law of conservation of energy**. An alternative form of KVL is *"Sum of voltage drops = Sum of voltage rises"*.




$$-v_1 + v_2 + v_3 - v_4 + v_5 = 0$$

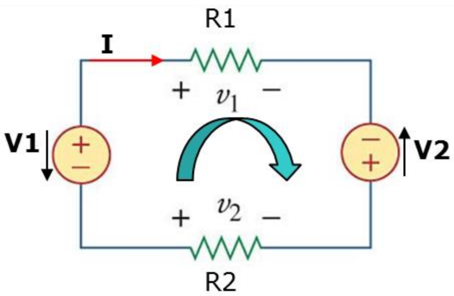
When voltage sources are connected in series, KVL can be applied to obtain the total voltage.




5. **EXERCISES**

All student have to do this part, after 15 minutes, student will be asked to resolve the problems.

**Example 1.**



Find $v_1$ and $v_2$ ; with: V1 = 10V, V2 = 8V; R1 = 4Ω; R2 = 2Ω

Solution:

- *Ohm's Law*:

$v1 = R1 * I = 4I$ and $v2 = R2 * I = 2I$

- *KVL:* clockwise around the loop, the sum of the voltage drops is:

$-V1 + v1 - V2 - v2 = 0$

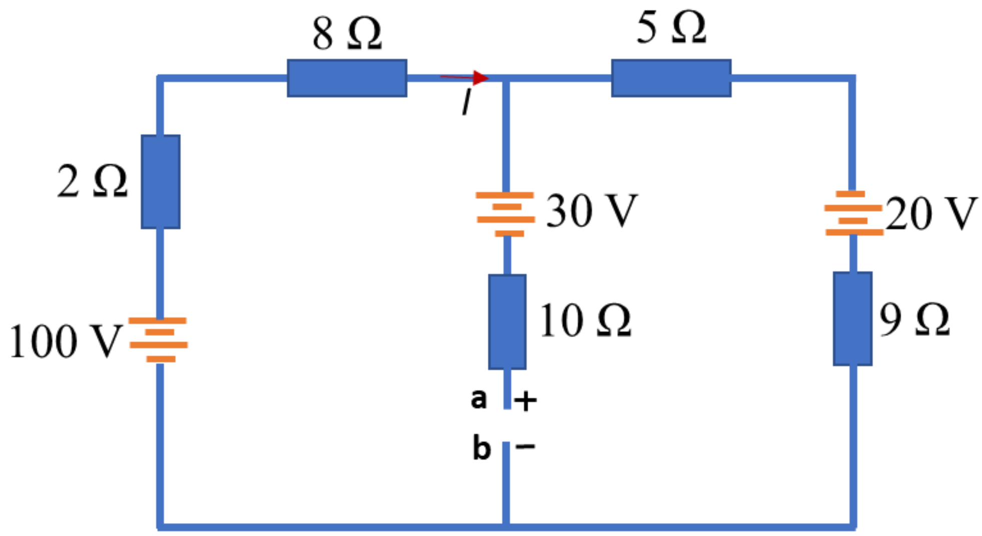
$-10 + 4 * I - 8 - 2 * I = 0$ → **I = 9A;** $v1 = 4I = 36V$ ; $v2 = 2I = 18V$

**Example 2.**

Calculate I and $V_{ab}$

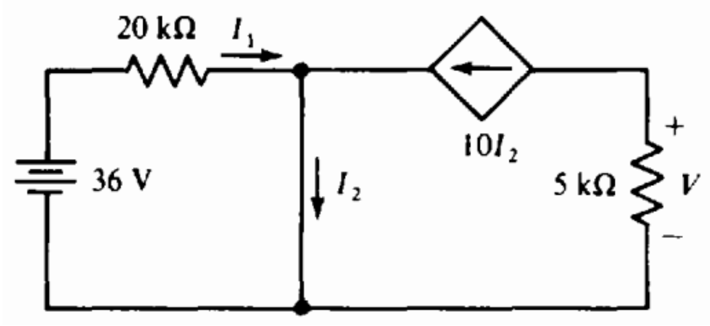
Because of the open circuit between nodes: **a** and **b**, the middle branch has no effect on the current I. Consequently, I can be obtained by applying KVL to the outside loop. The total resistance of this loop is: 2 + 8 + 5 + 9 = 24 Ω. And in the direction of I, the sum of the voltage rises from voltage sources is: 100 + 20 = 120 V. So, I = 120/24 = 5 A.

From the summing of voltage drops across the right-hand branch, the voltage drop, top to bottom across the middle branch is: 5(5) - 20 + 5(9) = 50 V. Consequently, Vab = 50 – 30 = 20 V; because there is zero volts across the 10Ω resistor.

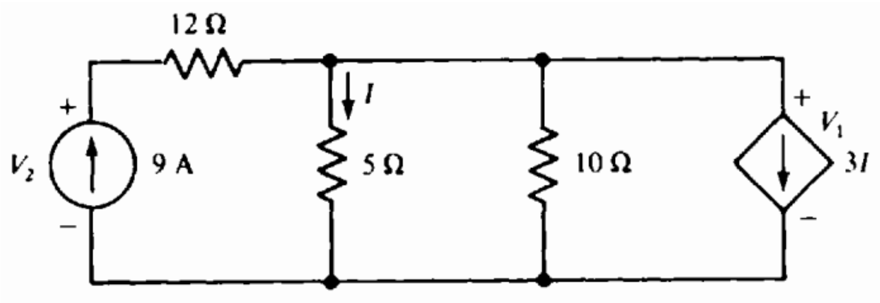


## 6. PROBLEMS (homework)

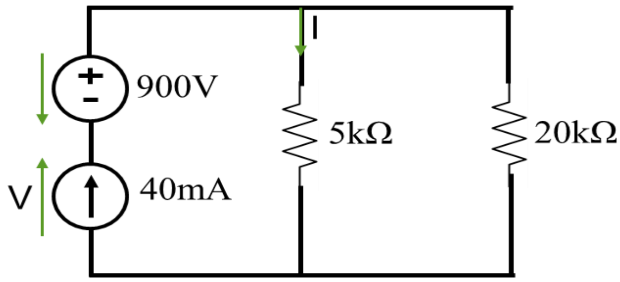
**Problem 1:** Find $I_1$, $I_2$ in this circuit:



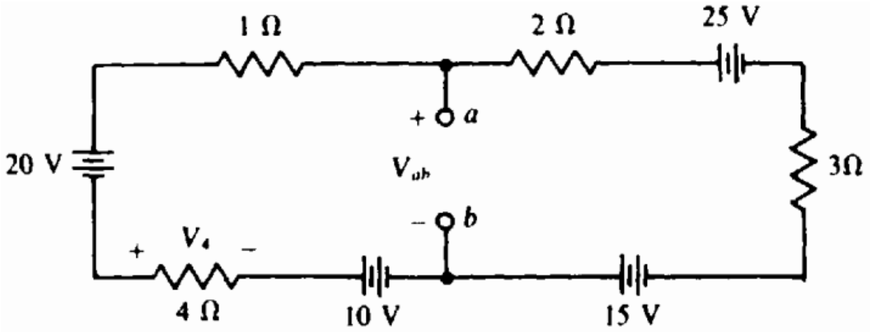

**Problem 2:** Find $V_1$; $V_2$ in this circuit:




**Problem 3:** Find I; V across the current source in this circuit:




**Problem 4:** Find $V_{ab}$ in this circuit:



# CHAPTER 2: DC CIRCUIT ANALYSIS TECHNIQUES

We have seen that using Kirchhoff's laws and Ohm's law we can analyze any circuit to determine the operating conditions (the currents and voltages). The challenge of formal circuit analysis is to derive the smallest set of simultaneous equations that completely define the operating characteristics of a circuit.

Here we apply the fundamental laws of circuit theory (Ohm's Law & Kirchhoff's Laws) to develop two powerful techniques for circuit analysis.

*** Nodal Analysis (based on KCL)**

*** Mesh/Loop Analysis (based on KVL)**

*This is the most important chapter for our course.*

## 1. NODAL ANALYSIS

- **Concept:**

- Developed based on the systematic approach of Kirchhoff's Current Law (KCL) to find all circuit variables without having to sacrifice any of the elements.

- General procedure which is making use of node voltages in circuit analysis as key solutions.

- **Importance terms**

- **Node Voltage**: Potential difference between a marked node and the selected reference node.

- **Element Voltage**: Potential difference across any element or branch in the circuit.

When Node Voltage = Element Voltage?

- **Why use Node Voltage?**

- Further reduce the number of equations to be solved simultaneously.

- No of independent equations = No of the marked nodes exclusive of the reference node.

- Element voltages and currents can be obtained in few steps using the solved node voltages.

- **NODAL ANALYSIS PROCEDURES: 8 STEPS (DETAIL) --- OR MAYBE 03 STEPS**

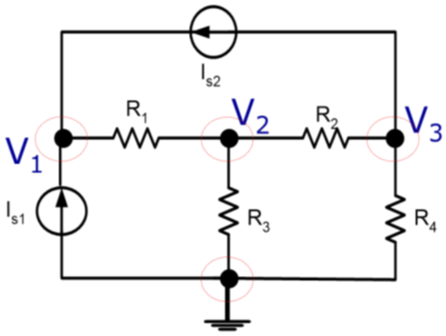
1) Clearly label all circuit parameters and distinguish the *unknown parameters from the known*.
2) Identify *all nodes* of the circuit
3) Select a *node as the reference node* also called the ground and assign to it a potential of 0 Volts. All other voltages in the circuit are measured with respect to the reference node.
4) Label the *voltages at all other nodes*.
5) Assign and *label polarities*.

6) Apply KCL at each non-reference nodes and express the branch currents in terms of the node voltages.
7) Solve the resulting *simultaneous equations for the node voltages*.
8) Now that the *node voltages are known*, the *branch currents* may be obtained from Ohm's law.

**EXERCISES**

**Example 1:**

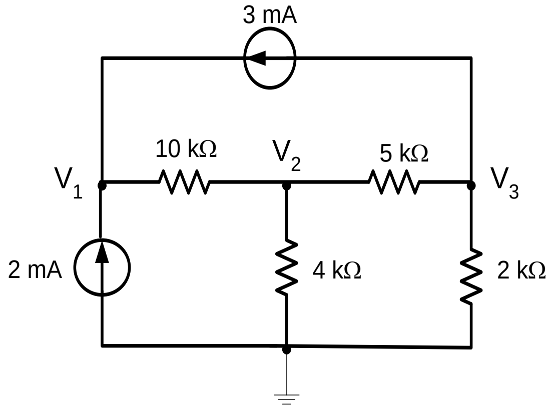
Applying Nodal analysis on simple circuit, find the power dissipated by the 10Ω resistor, Is2 = 3mA; R1=10k Ω; R2=5k Ω; R3=4k Ω; R4=2k Ω




**Solution:**

- Mark all essential nodes
- Assign unknown node voltages
- Indicate the reference node
- Perform KCL at each marked non-reference nodes using Ohm's law to formulate the equations in terms of the node voltages.

- KCL at $V_1$: $I_{s1} + I_{s2} = \dfrac{V_1 - V_2}{R_1}$ (1)
- KCL at $V_2$: $\dfrac{V_2 - V_1}{R_1} + \dfrac{V_2 - 0_1}{R_3} + \dfrac{V_2 - V_3}{R_2} = 0$ (2)
- KCL at $V_3$: $I_{s2} + \dfrac{V_3 - V_2}{R_2} + \dfrac{V_3 - 0}{R_4} = 0$ (3)




- KCL $V_1$: $2m + 3m = \frac{V_1 - V_2}{10k}$

  Simplify to $V_1 - V_2 = 50$ **(1)**

- KCL $V_2$: $\frac{V_2 - V_1}{10k} + \frac{V_2 - 0}{4k} + \frac{V_2 - V_3}{5k} = 0$

  Simplify to $-2V_1 + 11V_2 - 4V_3 = 0$ **(2)**

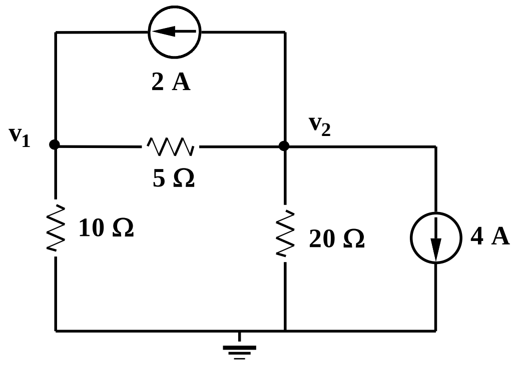
- KCL $V_3$: $3m + \frac{V_3 - V_2}{5k} + \frac{V_3 - 0}{2k} = 0$

  Simplify to $-2V_2 + 7V_3 = -30$ **(3)**

V1 = 60.55 [V]; V2 = 10.55 [V]; V3 = -1.27 [V]

The element voltage of R1 =10k Ω: ➔ $P_{R1} = \frac{(V1 - V2)^2}{R1} = \frac{(60.55 - 10.55)^2}{10} = 0.25[W]$

**Example 2:** Calculate the node voltages in the circuit below




**At $v_1$:**

$$\frac{V_1}{10} + \frac{V_1 - V_2}{5} = 2$$

**At $v_2$:**

$$\frac{V_2 - V_1}{5} + \frac{V_2}{20} = -6$$

$$3V_1 - 2V_2 = 20$$

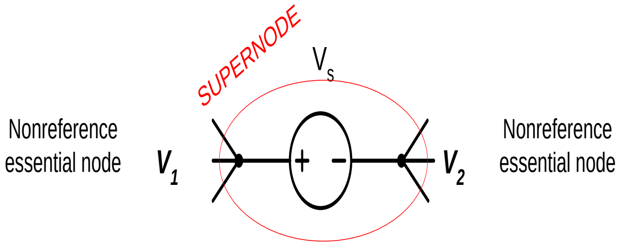
$$-4V_1 + 5V_2 = -120$$ **Solution:** $V_1 = -20$ V, $V_2 = -40$ V

**NOTE: APPLYING NODAL ANALYSIS ON CIRCUIT WITH VOLTAGE SOURCES**

*Three different effects depending on placement of voltage source in the circuit.*

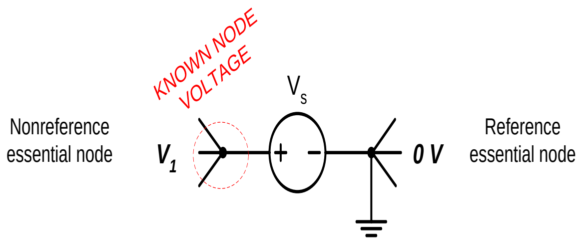
*Does the presence of a voltage source complicate or simplify the analysis?*

**Case 1: Voltage source between two non-reference essential nodes**



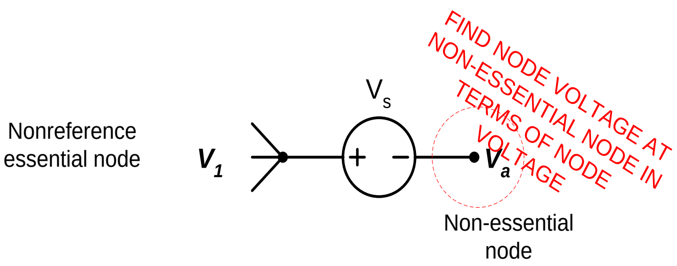

*Supernode Equation:* $V_s = V_1 - V_2$

**Case 2: Voltage source between a reference essential node and a non-reference essential node.**



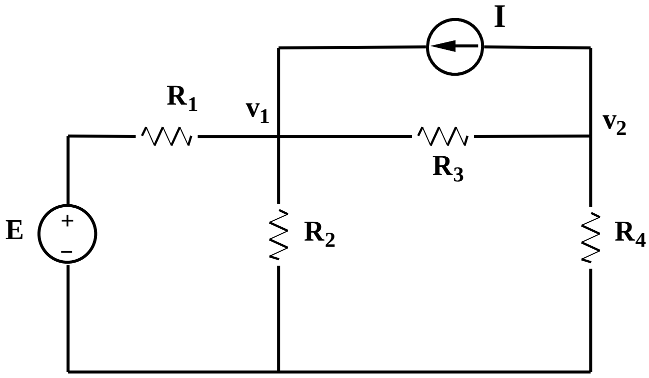

*Known node voltage:* $V_1 = V_s$

**Case 3: Voltage source between an essential node and a non-essential node.**




*Node voltage at non-essential node:* $V_a = V_1 - V_s$

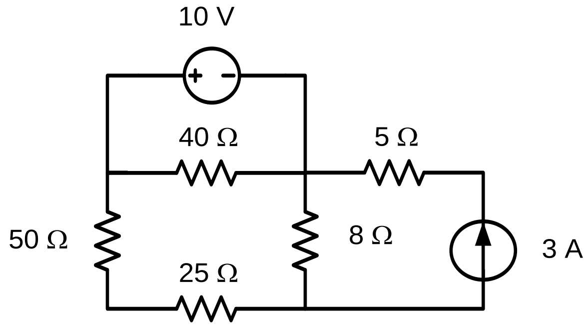
**Example 3: With voltage source**



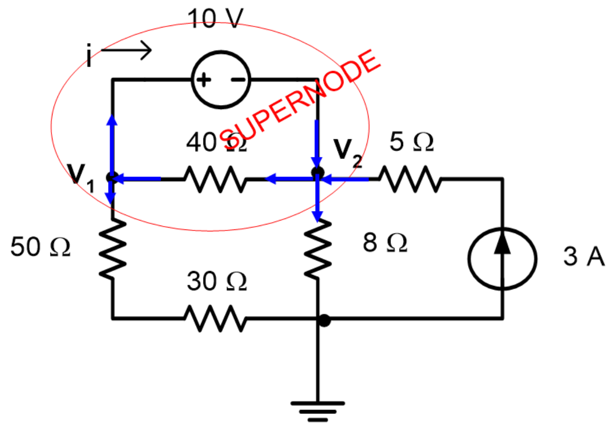

**At $V_1$:** $\frac{V_1 - E}{R_1} + \frac{V_1}{R_2} + \frac{V_1 - V_2}{R_3} = I$ $\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)V_1 - \left(\frac{1}{R_3}\right)V_2 = I + \frac{E}{R_1}$

**At $V_2$:** $\frac{V_2}{R_4} + \frac{V_2 - V_1}{R_3} = -I$ $-\left(\frac{1}{R_2}\right)V_1 + \left(\frac{1}{R_3} + \frac{1}{R_4}\right)V_2 = -I$

**Example 4:** Find the power of the 10-V voltage source? Is it supplying energy to the circuit or absorbing energy from the circuit? Show your work according to the nodal analysis procedure.




- Mark essential nodes and assign unknown node voltages and indicate the reference node




**- Perform KCL at each marked non-reference nodes using Ohm's law to formulate the equations in terms of node voltages.**

**At Node V1**

$$\frac{V_2 - V_1}{40} = i + \frac{V_1}{80}$$

**At Node V2:**

$3 + i = \frac{V_2}{8} + \frac{V_2 - V_1}{40}$ → $10V_2 + V_1 = 240$; and Super-Node Equation: $V_1 - V_2 = 10$

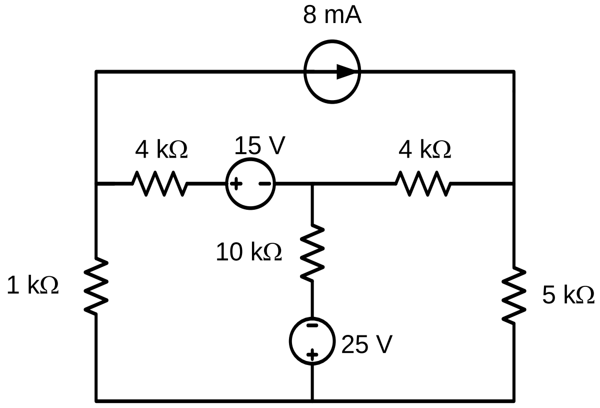
**We obtain:** $V_1 = 30.91$; $V_2 = 20.91$

Finding current through the voltage source: $\frac{V_2 - V_1}{40} = i + \frac{V_1}{80}$; $i = -0.636\,[A]$

Hence: $P_{voltagesource} = i * V = (-0.636) * 10 = -6.36\,[W]$, **negative sign → Delivering energy**

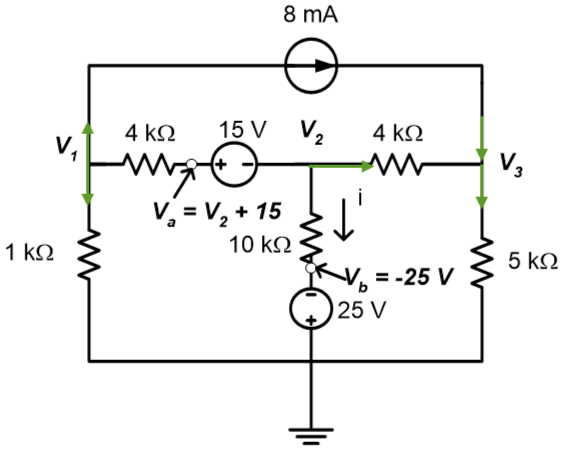
**Example 5:** Voltage source between an essential node and a non-essential node

Find the current through the 10 kΩ resistor. Show your work according to the nodal analysis procedure




Solution:

Mark essential nodes and assign *unknown node voltages* and indicate the *reference node*. For voltage sources, indicate the node voltages at both ends with respect to the assigned unknown node voltages at the essential nodes



- Perform KCL at each marked non-reference nodes using Ohm's law to formulate the equations in terms of node voltages

**At Node V1:**

$$\frac{V_1}{1k}+\frac{V_1-(V_2+15)}{4k}+8\,mA=0$$
$$\Rightarrow 5V_1-V_2=-17$$

**At Node V2**

$$\frac{V_2+25}{10k}+\frac{-V_1+(V_2+15)}{4k}+\frac{V_2-V_3}{4k}=0\,mA$$
$$\Rightarrow -10V_1+18V_2-10V_3=-250$$

**At Node V3**

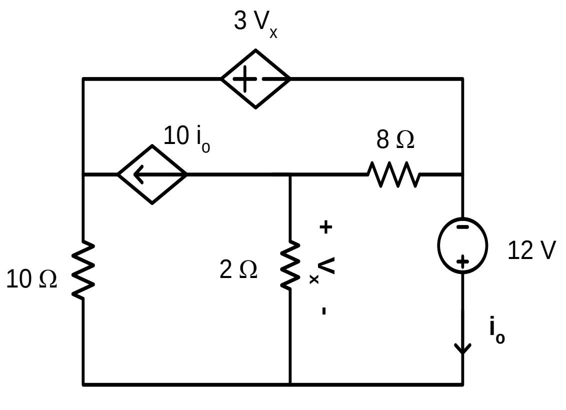
$$-\frac{V_3}{5k}+\frac{-V_3+V_2}{4k}+8\,mA=0$$
$$\Rightarrow -5V_2+9V_3=160$$

$$\Rightarrow V_1=-5.43\,;V_2=-10.17\,;V_3=12.13$$

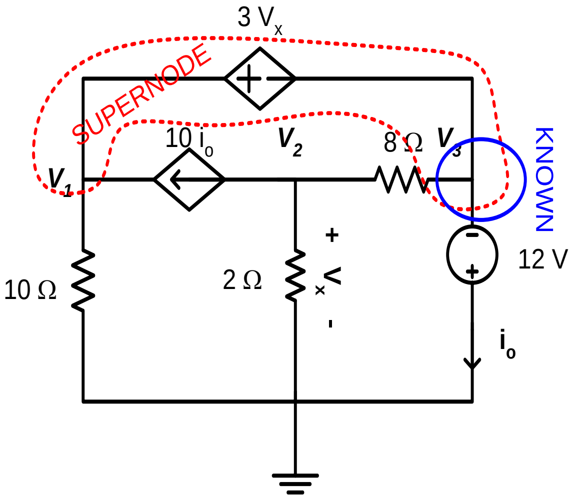
Finding current through the 10kΩ, KCL at $V_2$ : $i+\frac{V_2+15-V_1}{4k}+\frac{V_2-V_3}{4k}=0\,;\ i=3.01[mA]$

**Example 6: Circuit with dependent sources**

Use the node-voltage method to find both dependent terms $i_0$ and $V_x$ of the dependent sources of the circuit in Figure below:




- Mark essential nodes and assign unknown node voltages and indicate the reference node




- Perform KCL at each marked non-reference nodes using Ohm's law to formulate the equations in terms of node voltages.

$V_3 = -12V$

**KCL at Node V2**: $\frac{V_2}{2} + \frac{V_2 + 12}{8} = -10i_0$ ➔ $5V_2 + 80i_0 = -12$

Super-node equation: $3V_x = V_1 + V_3 = V_1 + 12$

Constraint equation: $V_x = V_2$; and $i_0 = \frac{V_2 + 12}{8} - \frac{V_1}{10} + 10i_0 \Rightarrow 720i_0 - 8V_1 + 10V_2 = -120$

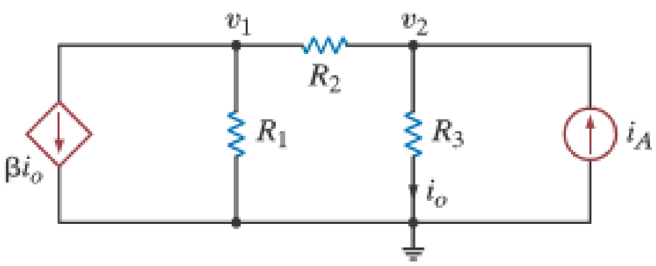
➔ $-V_1 + 3V_2 = 12$; $V_1 = -6.51$; $V_2 = 1.83$; $i_0 = -0.264$

**PROBLEM (homework)**

**Problem 1:**

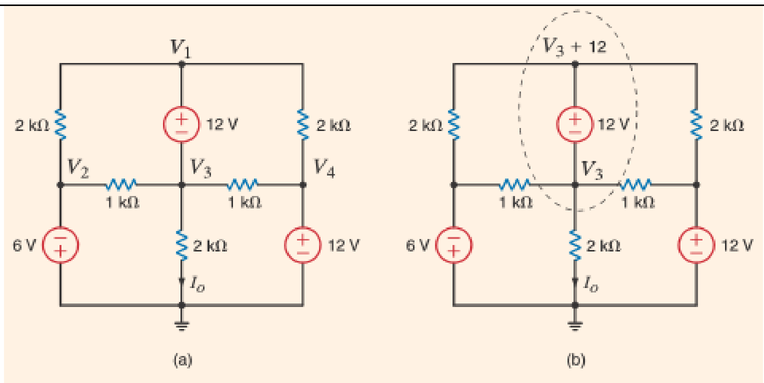
Determine the node voltages for this circuit, given the following parameters: $\beta = 2$;

$R_1 = 12k\Omega$; $R_2 = 6k\Omega$; $R_3 = 3k\Omega$; $i_A = 2mA$




**Problem 2:**

Let us determine the current $I_0$ in:




## 2. LOOP ANALYSIS

Loop analysis provides another general procedure for analyzing circuits, using loop/mesh currents as the circuit variables.

*Loop analysis is a special application of KVL on a circuit*. We use a special kind of loop called a 'mesh' which is a loop that does not have any other loops inside of it. A mesh starts at a node and traces a path around a circuit, returning to the original node without hitting any nodes more than once. We can only apply mesh analysis to planar circuits that is circuits without crossover connections. If a circuit cannot be redrawn without the intersecting disconnected lines then we cannot use mesh analysis.

### IMPORTANCE TERMS

**Mesh Current:** Assigned unknown current flows around the perimeter of the mesh/loop.

**Element Current:** Actual current through any element or branch in the circuit.

→ When Mesh Current = Element Current?

The mesh method uses the MESH CURRENTS AS THE CIRCUIT VARIABLES. The various steps are given below.

*1. Clearly label all circuit parameters and distinguish the unknown parameters from the known.*

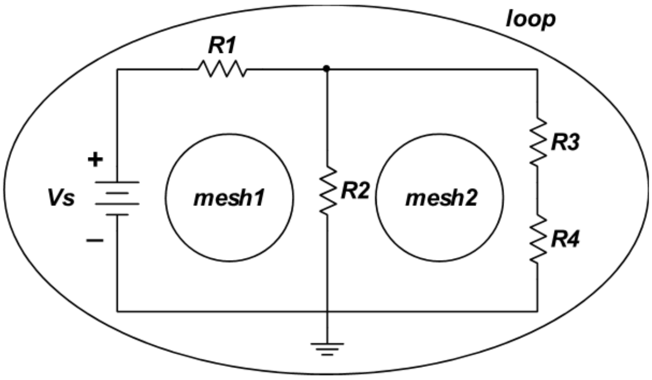
*2. Identify all meshes of the circuit.*

*3. Assign mesh currents and label polarities.*

*4. Apply KVL at each mesh and express the voltages in terms of the mesh currents.*

*5. Solve the resulting simultaneous equations for the mesh currents.*

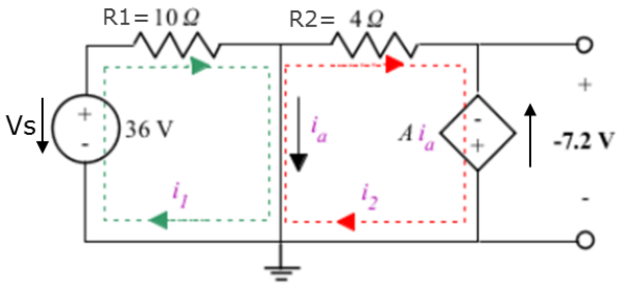
*6. Now that the mesh currents are known, the voltages may be obtained from Ohm's law.*




**EXERCISES**

**Example 1:**

Consider the following circuit. A voltmeter reads -7.2 volts across the dependent source. Find the gain A of the current-controlled voltage source.




Solution: First draw the meshes to be examined in the circuit. They are labelled in green with current $i_1$ and red $i_2$ with below.

Consider the dependent source. It is supplying a voltage in the opposite direction to the reading from the voltmeter. From this, we can infer the voltage of the source element:

$$Ai_a = -(-7.2) \Rightarrow Ai_a = 7.2\,[V]$$

The dependent current is between both meshes. We can represent this current in terms of the mesh currents:

$$i_a = i_1 - i_2$$

Now that we can represent the dependent source in terms of the mesh currents, apply KVL to obtain the equations. Mesh analysis on the **green loop** shows:

$$-V_s + R_1 i_1 = 0;\ -36 + 10i_1 = 0 \Rightarrow i_1 = 3.6\,[A]$$

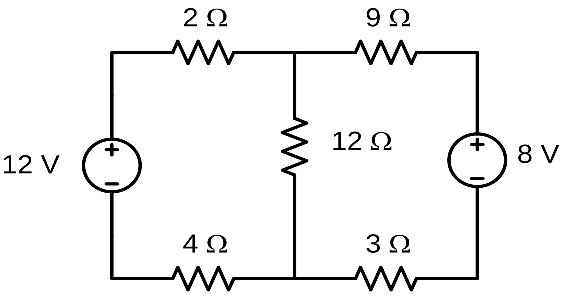
On the **second mesh** with the dependent source, the mesh equation will be:

$$-Ai_a + R_2 i_2 = 0;\ -7.2 + 4i_2 = 0 \Rightarrow i_2 = 1.8\,[A];$$

Using the values for and from the mesh equations, the dependent source gain can finally be found:

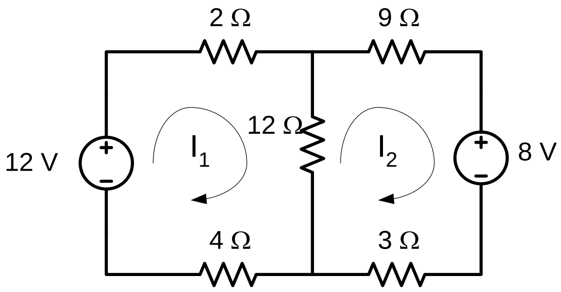
$$i_a = i_1 - i_2;\ i_a = 1.8\,[A];\ Ai_a = 7.2 \Rightarrow A = 4$$

**Example 2:**



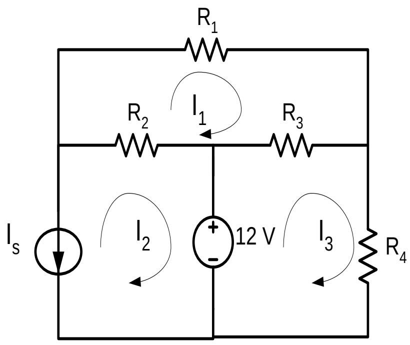

Find power dissipated in 12Ω-resistor and 3Ω-resistor using mesh analysis

<u>Solution:</u>



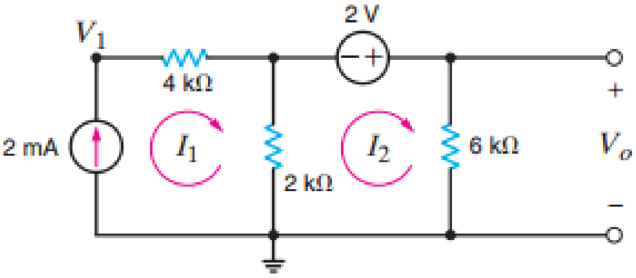

*2 meshes = 2 KVL Equations.*

**Case 1: Current source located at the outer most boundary**




- Connecting mesh current immediately known ($I_3$ = -$I_s$)
- No need to apply KVL around that loop/mesh.
- Mesh Current = Element Current = Current Source Value

**Example:** Let us find both $V_0$ and $V_1$ in the circuit in this circuit:



Although it appears that there are 02 unknown mesh currents, the current goes directly through the current source and, therefore, is constrained to be $I_1 = 2\,mA$. Hence, only the current is unknown. KVL for the rightmost mesh $I_2$ is:

$2*(I_2 - I_1) - 2 + 6*I_2 = 0;\ I_1 = 3/4\,mA$

$V_0 = I_2 * 6 = 9/2\,[V]$

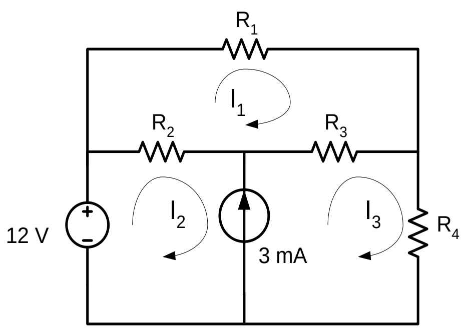
To obtain $V_1$ (voltage across the current source), we apply KVL around any closed path. If we use the outer loop, the KVL equation is:

$V_1 - 4*I_1 - 2 + 6*I_2 = 0;\ V_1 = 21/2\,[V]$

**Case 2: Current source located at the boundary between 2 meshes**

- Enclose the current source and combine the two meshes to form a SUPER-MESH.
- KVL is performed around the super-mesh; do not consider voltage across current source.
- Formulate simultaneous mesh equation – express the relationship between mesh currents that form the s/mesh and current source that it encloses

**Example:**



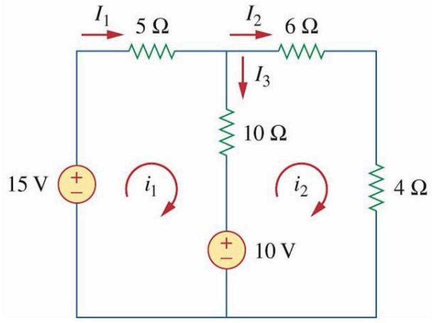

KVL for loop ($I_2$ and $I_3$): $-12 + (I_2 - I_1)*R_2 + (I_3 - I_1)*R_3 + I_3 * R_4 = 0$

KVL for loop ($I_1$): $R_1 * I_1 + (I_1 - I_2)*R_2 + (I_1 - I_3)*R_3 = 0$

Super-Mesh: $I_3 - I_2 = I_s$

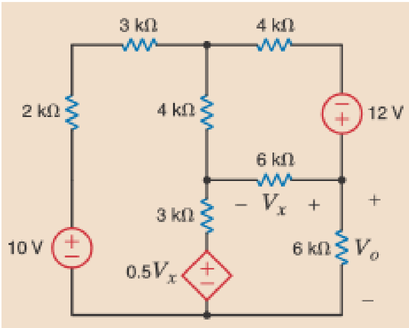
**Example 3:**

Find the branch currents $I_1$, $I_2$, and $I_3$, $V_x$ using mesh analysis




**Example 4:**

Find $V_0$: using mesh analysis.




**Example 5:**

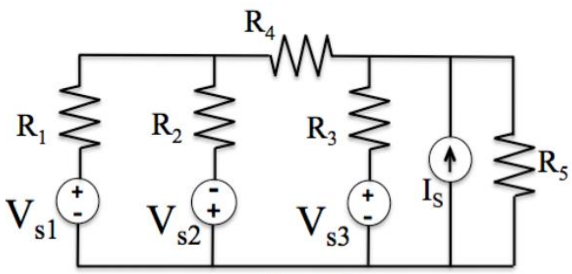


Figure 1

Consider the circuit shown in Figure 1

a) Write the nodal equations describing the circuit.

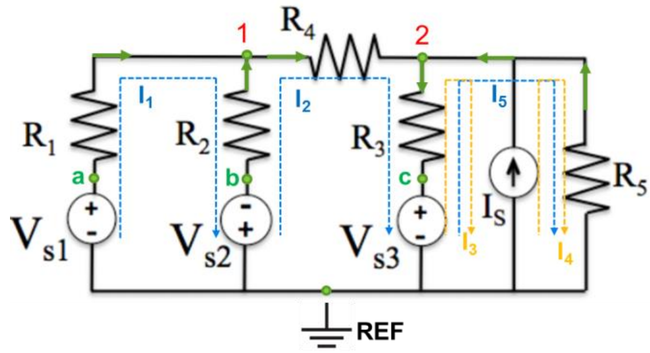
b) Determine the branch currents by nodal voltages.

c) Write loop equations describing the circuit.

d) Determine the branch currents by loop currents.

e) Give Vs1 = 100V, Vs2 = 80V, Vs3 = 24V, Is = 4A, R1 = 5 Ω, R2 = 15 Ω, R3 = 4 Ω, R4 = 8 Ω, R5 = 6 Ω. Determine the voltages across R3, R4 and R5.

**Solution:**




Note that:

$V_a = V_{S1}\,; V_b = -V_{S2}\,; V_c = V_{S3}\,;$

a) Write the nodal equations describing the circuit + b) Determine the branch currents by nodal voltages.

Mark essential nodes and assign unknown node voltages $V_1; V_1$; reference node.

Ohm's law:

$$I_{R1} = \frac{V_a - V_1}{R_1} = \frac{V_{S1} - V_1}{R_1}\,;\ I_{R2} = \frac{-V_1 + V_b}{R_2} = \frac{-V_1 - V_{S2}}{R_2};\ I_{R4} = \frac{V_1 - V_2}{R_4};\ I_{R3} = \frac{V_2 - V_c}{R_3} = \frac{V_2 - V_{S3}}{R_2}$$

Nodal Equations, apply KCL:

**- At node 1:**

$$I_{R1} + I_{R2} - I_{R4} = 0;\ \frac{V_{S1} - V_1}{R_1} + \frac{-V_1 - V_{S2}}{R_2} - \frac{V_1 - V_2}{R_4} = 0 \quad \rightarrow$$

$$\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_4}\right) * V_1 - \frac{1}{R_4} * V_2 = V_{S1} * \frac{1}{R_1} + V_{S2}\frac{1_2}{R_2}\,; \qquad 40V_1 - 15V_2 = 3040 \qquad (1)$$

**- At node 2:**

$$I_{R4} + I_{R5} + I_S - I_{R3} = 0;\ \frac{V_1 - V_2}{R_4} - \frac{V_2}{R_5} + I_S - \frac{V_2 - V_{S3}}{R_2} = 0\,;$$

$$\frac{1}{R_4} * V_1 - \left(\frac{1}{R_5} + \frac{1}{R_2} + \frac{1}{R_4}\right) * V_2 = -I_S - V_{S3}\frac{1_2}{R_2}; \qquad 15V_1 - 43V_2 = -672 \qquad (2)$$

From (1) and (2), we obtain: $V_1 = 46.78V\,; V_2 = 29.26$

$I_{R1} = 10.64A;\ I_{R2} = -8.52A;\ I_{R3} = 1.32A;\ I_{R4} = 2.19A;\ I_{R2} = -4.88A$

c) Write loop equations describing the circuit.

First draw the meshes to be examined in the circuit. They are labelled in figure with current $I_1$; $I_2$; $I_3$; $I_4$; $I_5$ with below.

Loop Equations, apply KVL:

Loop 1: $(R_1 + R_2) * I_1 - R_2 * I_2 = V_{S2} + V_{S1}$

Loop 2: $(-R_2) * I_2 + (R_2 + R_3 + R_4) * I_2 - R_3 * I_3 = -V_{S3} - V_{S2}$

Super-Mesh: $I_4 - I_3 = I_S$

Note that: voltage across the current source $V_x$, can determine:

Loop 3: $-V_{S3} - R_3 * I_2 + R_3 * I_3 - V_x = 0$, and Loop 4: $R_5 * I_4 + V_x = 0$

So: $-R_3 * I_2 + R_3 * I_3 + R_5 * I_4 = V_{S3}$

→ $I_{R1} = I_1 = 10.64A$; $I_{R2} = I_2 - I_1 = -8.5A$; $I_{R4} = I_2 = 2.19A$; $I_{R3} = I_2 - I_5 = 1.3A$

**Example 6**:Example 3.16, page 128, [1]

**NOTE:**

(a) Nodal analysis applies KCL to find unknown voltages in a given circuit, while mesh analysis applies KVL to find unknown currents.

(b) Using mesh currents instead of element currents as circuit variables is convenient and reduces the number of equations that must be solved simultaneously.

(c) Mesh analysis is not quite as general as nodal analysis because it is only applicable to a circuit that is planar. A planar circuit is one that can be drawn in a plane with no branches crossing one another; otherwise it is nonplanar.

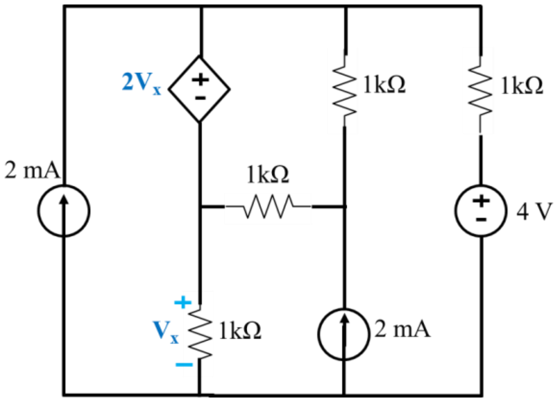
(d) Given a network to be analyzed, how do we know which method is better or more efficient? Suggestion: Choose the method that results in smaller number of variables or equations.

• *A circuit with fewer nodes than meshes is better analyzed using nodal analysis, while a circuit with fewer meshes than nodes is better analyzed using mesh analysis.*

**PROBLEM (homework)**

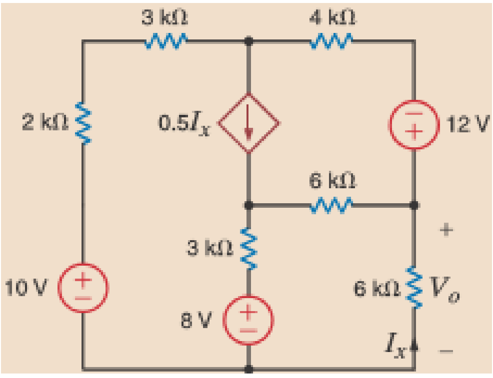
**Problem 1:**

Let us find $V_x$ in the network in:




**Problem 2:**

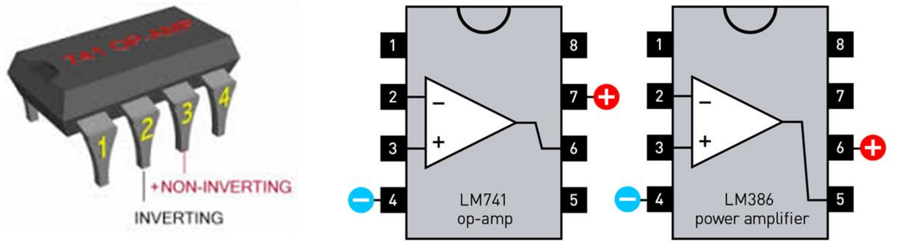
Let us find all parameters: $I_x$ ; $V_0$ in the network in:



# CHAPTER 3: OPERATIONAL AMPLIFIERS CIRCUITS

## 1. INTRODUCTION

Operational Amplifiers: electronic device that can amplify (increase/ decrease) the amplitude of signal. They are represented both schematically and realistically below, they are integrated circuits (ICs):




- An op-amp is a high *voltage gain*, DC amplifier with high input impedance, low output impedance, and differential inputs.
- *Positive input at the non-inverting input produces positive output, positive input at the inverting input produces negative output.*

**Notation: The basic op amp with supply voltage included is shown in the diagram below.**

Where:

$V_+$ : non-inverting input

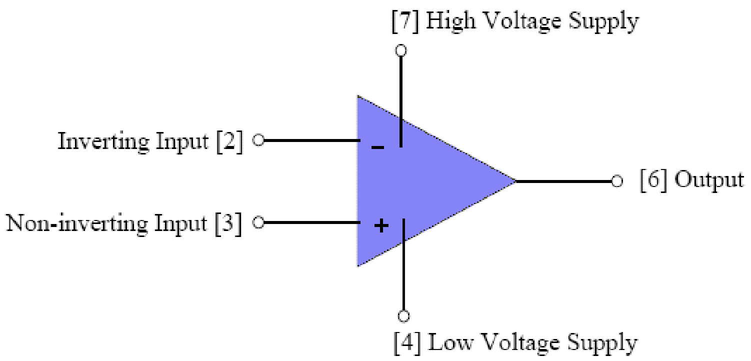
$V_-$ : inverting input

$V_{out}$: output

$V_{S+}$: positive power supply (sometimes also $V_{DD}$, $V_{CC}$, or $V_{CC+}$ )

$V_{S-}$ : negative power supply (sometimes also $V_{SS}$, $V_{EE}$, or $V_{CC-}$ )

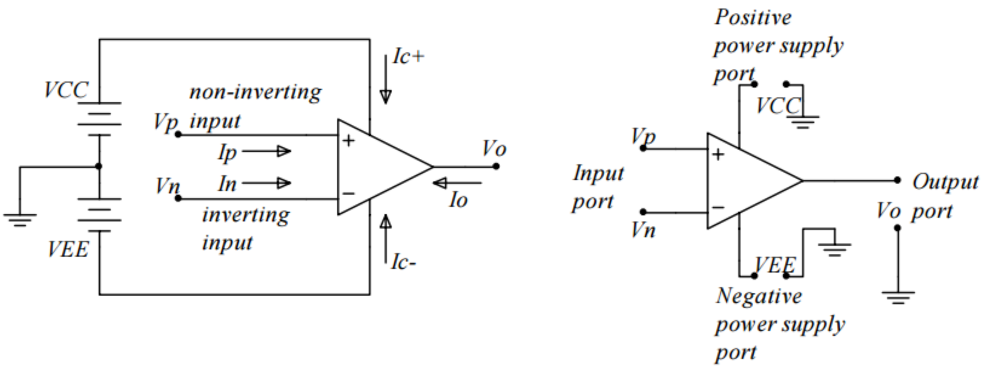
The power supply pins ($V_{S+}$ and $V_{S-}$) can be labeled in different ways




+ An operational amplifier is modeled as a voltage controlled voltage source.

+ An operational amplifier has a very high input impedance and a very high gain

The operational amplifier (op-amp) is a voltage controlled voltage source with very high gain. It is a five terminal four port active element. The symbol of the op-amp with the associated terminals and ports is shown:

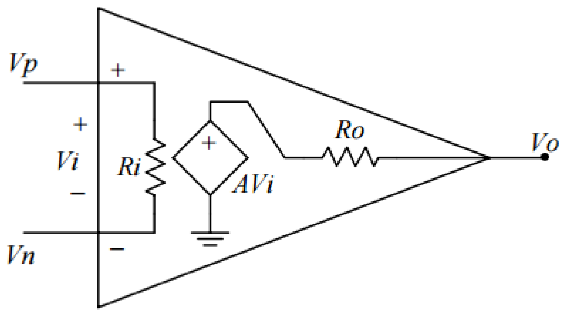



The power supply voltages $V_{CC}$ and $V_{EE}$ power the operational amplifier and in general define the output voltage range of the amplifier. The terminals labeled with the "+" and the "-" signs are called non-inverting and inverting respectively. The input voltage $V_p$ and $V_n$ and the output voltage $V_o$ are referenced to ground. The five terminals of the op-amp form one (complicated) node and if the currents are defined as shown on figure, the KCL requires that:

$$I_n + I_p + I_{c-} + I_0 = 0$$

Therefore, for current balance we must include all currents. *This is what defines an active element*. If we just consider the signal terminals, then there is no relationship between their currents. In particular: $I_n + I_p + I_0 \# 0$

**The equivalent circuit model of an op-amp** is shown on:




The voltage $V_i$ is the differential input voltage: $V_i = V_p - V_n$. $R_i$ is the input resistance of the device and $R_o$ is the output resistance. The gain parameter A is called the open loop gain. *The open loop configuration of an op-amp is defined as an op-amp circuit without any circuit loops that connect the output to any of the inputs.*

In the absence of any load at the output, the output voltage is: $V_0 = AV_i = A(V_p - V_n)$

Which indicates that the output voltage $V_o$ is a function of the difference between the input voltages $V_p$ and $V_n$. For this reason, op-amps are difference amplifiers.

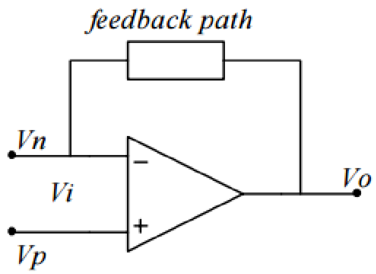
**Use of OP-AMP**

+ Op amps can be configured in many ways using resistors and other components.

+ Operational amplifiers are almost always used with negative feedback, in which part of the output signal is returned to the input in opposition to the source signal.

+ Amplifiers provide gains in voltage or current.

+ Op amps can convert current to voltage

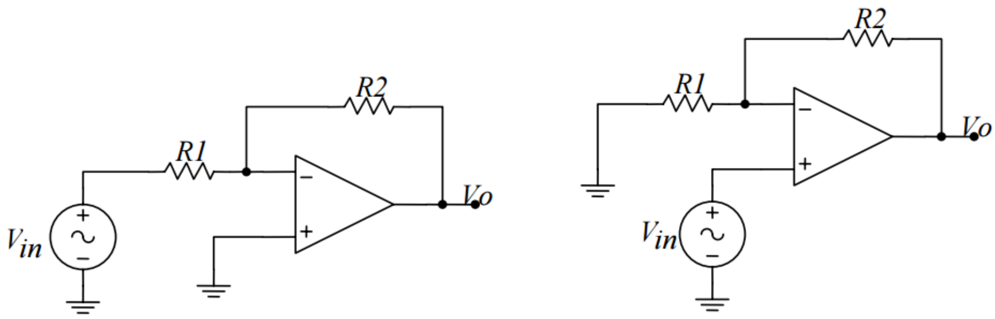



**Building Negative Feedback Amplifiers**

With two resistors we can construct the fundamental feedback network of a negative feedback amplifier. Depending on the terminal at which the signal is applied, the fundamental negative feedback configuration can be:

- in the inverting amplifier arrangement, where the input signal, $V_{in}$, is applied to the inverting terminal,

- or in the non-inverting amplifier arrangement, where the input signal, $V_{in}$, is applied to the non-inverting terminal.

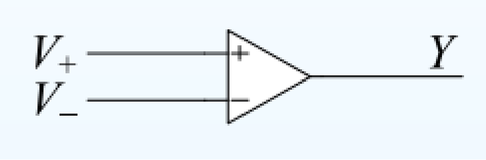



(a) Inverting amplifier (b) Non-inverting amplifier

We will perform the analysis by considering both the effect of finite open loop gain A is finite; and the ideal op-amp model for which $A \to \infty$.

## 2. ANALYSING OP-AMP CIRCUITS

### a. Nodal analysis is simplified by making some assumptions




1. **Check for negative feedback**: to ensure that an increase in Y makes ($V_+ - V_-$) decrease, Y must be connected (usually via other components) to $V_-$, example: resistance
2. Assume V+ = V−: Since (V+ − V−) = Y/A, this is the same as assuming that A = ∞. →Requires negative feedback.
3. Assume zero input current: in most circuits, the current at the op-amp input terminals is much smaller than the other currents in the circuit, so we assume it is zero.
4. Apply KCL at each op-amp input node separately (input currents = 0).
5. Do not apply KCL at output node (output current is unknown).

**Note:** The op-amp needs two power supply connections; usually +15 V and −15 V. These are almost always omitted from the circuit diagram. The currents only sum to zero (KCL) if all five connections are included:

$V_{S+}$: positive power supply (sometimes also $V_{DD}$, $V_{CC}$, or $V_{CC+}$ )

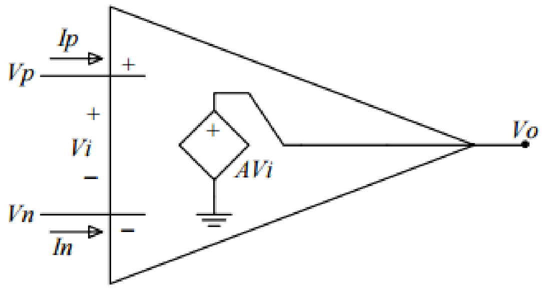
$V_{S-}$ : negative power supply (sometimes also $V_{SS}$, $V_{EE}$, or $V_{CC-}$ )

### b. Ideal Op-amp

From a practical point of view, an ideal op-amp is a device which acts as an ideal voltage controlled voltage source. The device will have the following characteristics:

- No current flows into the input terminals of the device. This is equivalent to having an infinite input resistance: $R_i = \infty$. In practical terms this implies that the amplifier device will make no power demands on the input signal source.
- Have a zero output resistance: $R_0 = 0$. This implies that the *output voltage is independent of the load connected to the output*.

In addition the ideal op-amp model will have infinite open loop gain (A → ∞). The ideal op-amp model is shown schematically on:




In summary, the ideal op-amp conditions are:

$I_n = I_p = 0$ : No current into the input terminals

$R_i = \infty$ : Infinite input resistance

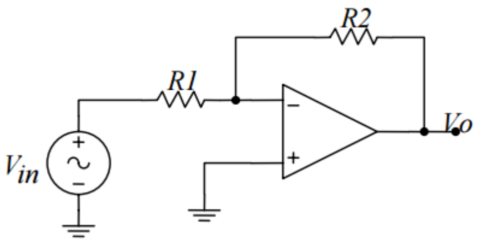
$R_0 = 0$ : Zero output resistance

$A \rightarrow \infty$: Infinite open loop gain

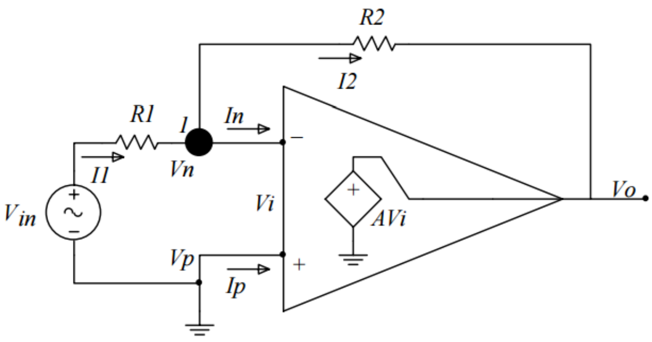
## 3. PUPULART OP-AMP CIRCUITS

### 3.1. Inverting Amplifiers

The basic inverting amplifier configuration is shown on this figure. The input signal ($V_{in}$) is applied to the inverting terminal and the balance of the circuit consists of resistors R1 and R2.




Let's analyze this circuit, i.e determine the output voltage $V_0$ as a function of the input voltage $V_{in}$ and the circuit parameters, by assuming infinite input resistance at the inverting and non-inverting terminals, zero output resistance and finite open loop gain A. The equivalent circuit of this model is shown on:



**Example:**

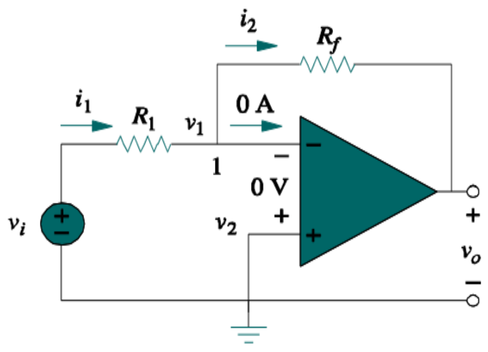
**Ex 1.**

Since the noninverting terminal is grounded

$$v_1 = v_2 = 0$$

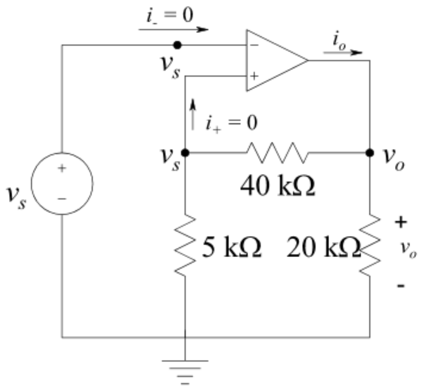
KCL at $v_1$: $i_1 = i_2 \Rightarrow \frac{v_i - 0}{R_1} = \frac{0 - v_0}{R_f}$

$$v_0 = -\frac{R_f}{R_1} v_i$$




Gain, $G = -\frac{R_f}{R_1}$

**Ex 2.**




KCL at noninverting terminal:

$$\frac{v_s - v_0}{40} + \frac{v_s}{5} = 0$$

$$9v_s = v_0$$

KCL at $v_0$:

$$i_0 = \left(\frac{v_0}{20k} + \frac{v_0 - v_s}{40k}\right)$$

If $v_s$ = 1 V then $i_0 = 0.65$ mA

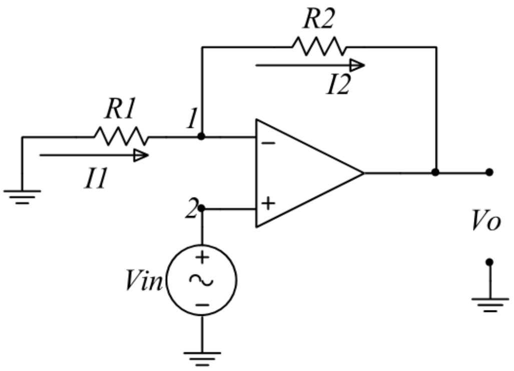
**Inverting Amplifier. Ideal op-amp circuit analysis**

The ideal op-amp rules are:

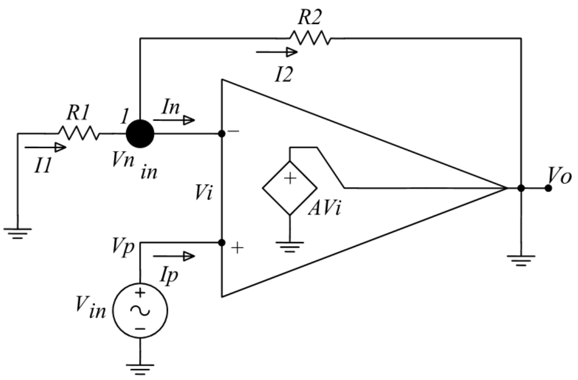
1. The differential input voltage is zero. — $Vi = 0 \rightarrow Vn = Vp$
2. No current flowing into the input terminals. This is equivalent to infinite input resistance for the op-amp Ri = ∞ — $In = Ip = 0$
3. Infinite open loop gain. — $A \rightarrow \infty$
4. Output resistance is zero — $Ro = 0$

### 3.2. Non-Inverting Amplifier

This picture shows the basic non-inverting amplifier configuration. The negative feedback is maintained, and the input signal is now applied to the non-inverting terminal.




The equivalent circuit of the Non-Inverting amplifier with a finite open-loop gain is shown on this figure. Here we have assumed an infinite input resistance and a zero output resistance for the op-amp.




Since $In = Ip=0$, we have $I1=I2$ and therefore:

$$\frac{-Vn}{R1} = \frac{Vn - Vo}{R2} \Rightarrow \frac{Vo}{R2} = Vn\left(\frac{1}{R1} + \frac{1}{R2}\right)$$

Since the voltage $Vi = Vp\text{-}Vn = Vin\text{-}Vn,$ the output voltage is given by:

$$V_0 = A(V_{in} - V_n)\text{; with } G = \frac{V_0}{V_{in}} = \frac{1 + \dfrac{R2}{R1}}{1 + (1 + \dfrac{R2}{R1})/A},$$

The gain is positive and unlike the inverting amplifier, the output voltage $V_0$ is in phase with the input Vin and the gain is always greater than 1.

**Non-inverting amplifier: Ideal model**

The ideal model implies the voltages at nodes 1 and 2 are equal: $V_n = V_{in}$. Also, since no current flows into the terminals of the op-amp, KCL at node 1 gives,

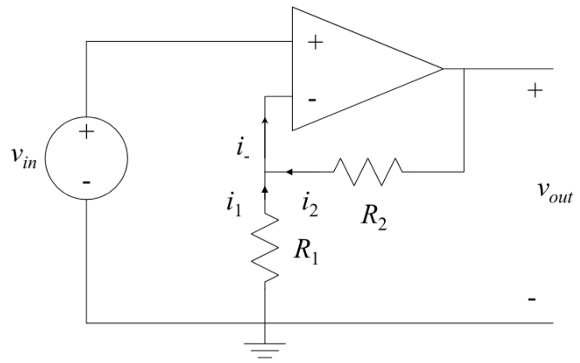
$$\left.\begin{aligned} I1 &= I2 \\ I1 &= -\frac{Vin}{R1} \\ I2 &= \frac{Vin - Vo}{R2} \end{aligned}\right\} \Rightarrow -\frac{Vin}{R1} = \frac{Vin - Vo}{R2}$$

$$G = \frac{V_0}{V_{in}} = 1 + \frac{R2}{R1}$$

**Example:**

$$i_- = 0$$

$$i_1 = \frac{-v_-}{R_1} = \frac{-v_{in}}{R_1}$$ Since $v_- = v_+ = v_{in}$

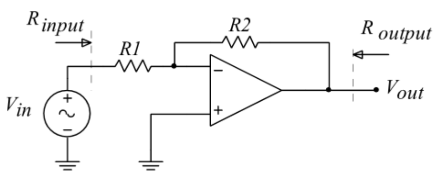
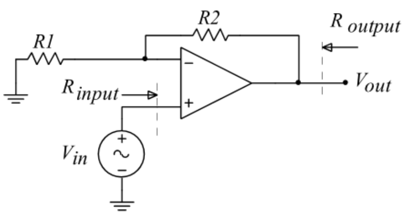



$$i_2 = \frac{v_{out} - v_-}{R_2} = \frac{v_{out} - v_{in}}{R_2}$$

$$\frac{-v_{in}}{R_1} + \frac{v_{out} - v_{in}}{R_2} = 0$$

$$\boxed{v_{out} = v_{in}\left(1 + \frac{R_2}{R_1}\right)}$$

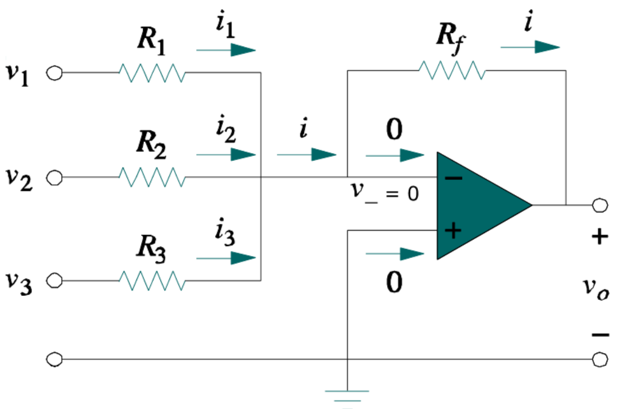
## SUMMARY OF IDEAL AMPLIFIER CHARACTERISTICS

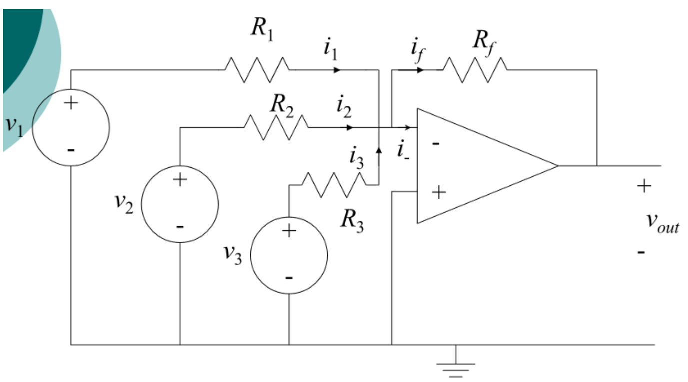
| | Inverting amplifier  | Non inverting amplifier  |
|---|---|---|
| Gain: $\left(\frac{Vout}{Vin}\right)$ | $-\frac{R2}{R1}$ | $1+\frac{R2}{R1}$ |
| Rinput | $R1$ | $\infty$ |
| Routput | 0 | 0 |



## 4. CIRCUITS WITH MULTIPLE OPERATIONAL AMPLIFIERS

### 1. Inverting Summer



**KCL at non-inverting node:**

$$i_1 = \frac{v_1 - v_-}{R_1} = \frac{v_1}{R_1} \quad \text{since } v_- = 0$$

$$i_2 = \frac{v_2 - v_-}{R_2} = \frac{v_2}{R_2}$$

$$i_3 = \frac{v_3 - v_-}{R_3} = \frac{v_3}{R_3}$$

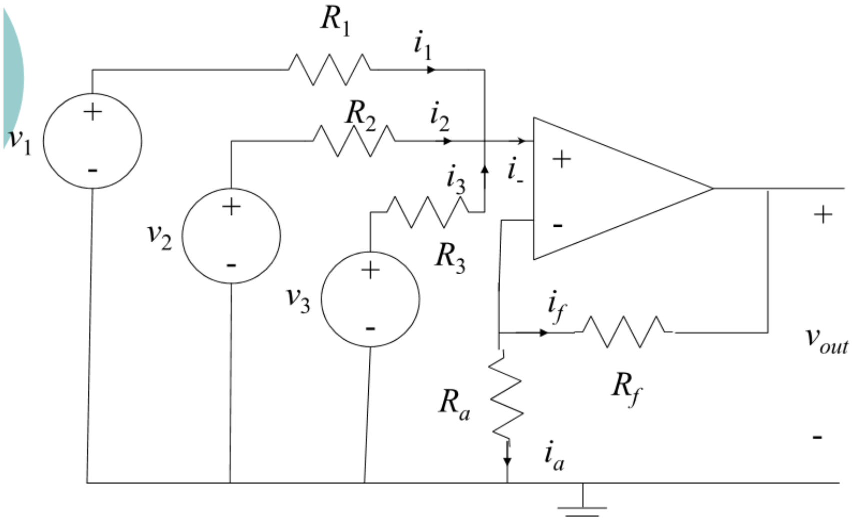
$$i_- = 0$$

$$i_f = \frac{v_{out} - v_-}{R_f} = \frac{v_{out}}{R_f}$$

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} + \frac{v_{out}}{R_f} = 0$$

$$v_{out} = -\frac{R_f}{R_1}v_1 - \frac{R_f}{R_2}v_2 - \frac{R_f}{R_3}v_3$$

**2. Non-inverting Summer**



KCL at noninverting input:

$$i_1 + i_2 + i_3 = 0$$

$$i_1 = \frac{v_1 - v_+}{R_1}$$

$$i_2 = \frac{v_2 - v_+}{R_2}$$

$$i_3 = \frac{v_3 - v_+}{R_3}$$

KCL at inverting input:

$$i_f + i_a = 0$$

$$i_f = \frac{v_- - v_{out}}{R_f}$$

$$i_a = \frac{v_-}{R_a}$$

$$v_- = \frac{R_a}{R_a + R_f} v_{out}$$

$$v_- = v_+$$

$$i_1 + i_2 + i_3 = 0$$

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) v_+$$

$$\frac{1}{R_T} = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{v_1}{R_1} + \frac{v_2}{R_2} + \frac{v_3}{R_3} = \frac{1}{R_T} \frac{R_a}{R_a + R_f} v_{out}$$

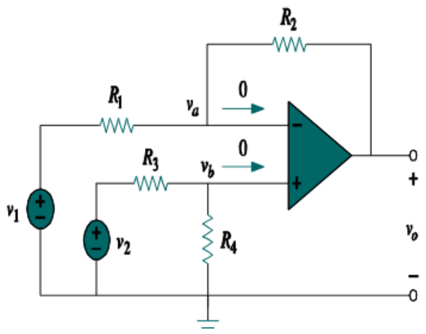
$$v_{out} = \left(1 + \frac{R_f}{R_a}\right)\left(\frac{R_T}{R_1} v_1 + \frac{R_T}{R_2} v_2 + \frac{R_T}{R_3} v_3\right)$$

### 3. The difference amplifier

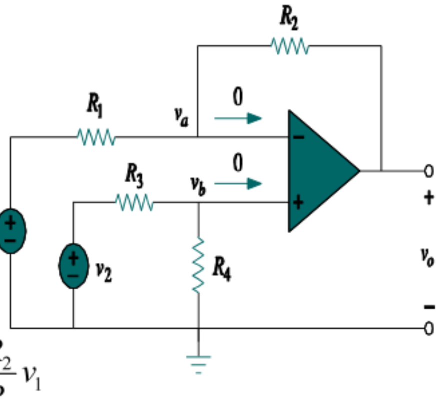
KCL at node $v_b$ :

$$\frac{v_b - v_2}{R_3} = \frac{v_b}{R_4}$$

$$v_b = \frac{R_4}{R_3 + R_4} v_2 = v_a$$



KCL at $v_a$




$$\frac{v_a - v_1}{R_1} + \frac{v_a - v_o}{R_2} = 0$$

$$\frac{1}{R_2} v_o = \left( \frac{1}{R_1} + \frac{1}{R_2} \right) v_a - \frac{1}{R_1} v_1$$

$$v_o = \left( 1 + \frac{R_2}{R_1} \right) v_a - \frac{R_2}{R_1} v_1 = \left( 1 + \frac{R_2}{R_1} \right) \frac{R_4}{R_3 + R_4} v_2 - \frac{R_2}{R_1} v_1$$

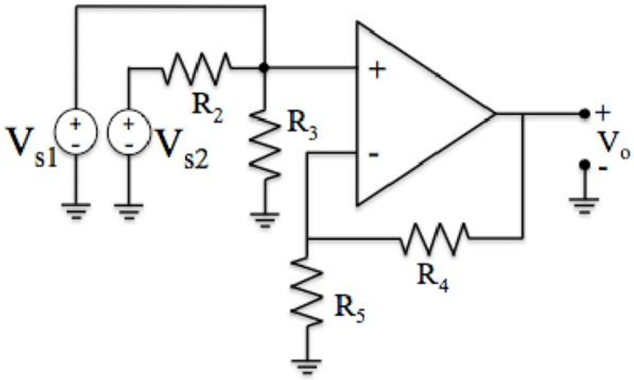
$$v_0 = \left( 1 + \frac{R_2}{R_1} \right) \frac{1}{\frac{R_3}{R_4} + 1} v_2 - \frac{R_2}{R_1} v_1 = \frac{R_2}{R_1} \frac{\left( \frac{R_1}{R_2} + 1 \right)}{\frac{R_3}{R_4} + 1} v_2 - \frac{R_2}{R_1} v_1$$

Since a difference must reject a signal common to the two inputs, the amplifier must have the property that $V_0 = 0$ when $V_1 = V_2$ . This implies that:

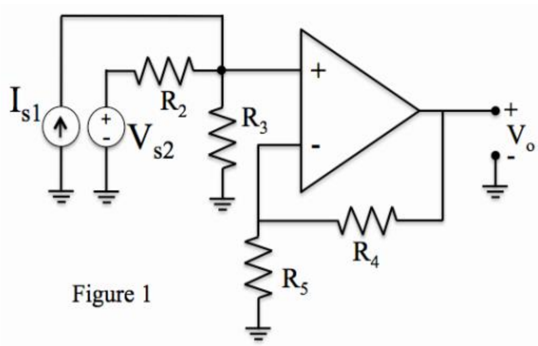
$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

## 5. PROBLEMS

**Ex1.** Derive an expression for $V_0$ in terms of $V_{S1}$ and $V_{S2}$

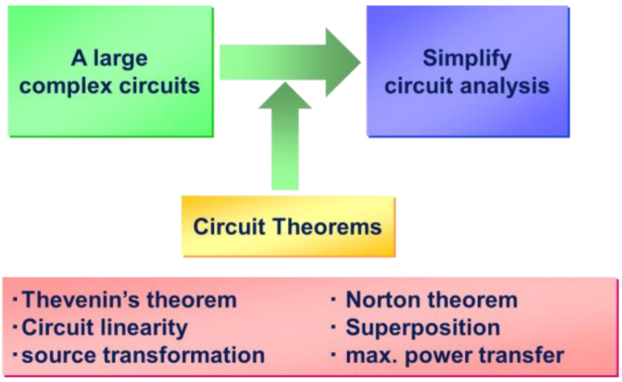



**Ex2.** Derive an expression for $V_0$ in terms of $I_{S1}$ and $V_{S2}$



# CHAPTER 4: NETWORK THEOREMS

## 1. INTRODUCTION




**Network Theorems:**

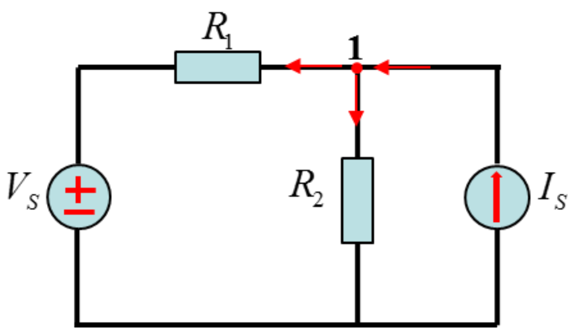
- The fundamental laws that govern electric circuits are the Ohm's Law and the Kirchoff's Laws.

- Technique to analyze a circuit: Node and Mesh

**Linearity**

**Example:**

Suppose we use variables instead of fixed values for all of the independent voltage and current sources. We can then use nodal analysis to find all node voltages in terms of the source values.




Write node equation: node voltage at node 1: $V$

$$-\frac{V-V_s}{R_1}-\frac{V}{R_2}+I_S=0$$

$(\frac{1}{R_1}+\frac{1}{R_2})V=\frac{1}{R_1}V_S+I_S$; we have two kind of sources: $V_S$, $I_S$

$V$, $R_1$, $R_2$, ...linearity; → $V=a_1V_1+a_2V_2+...+b_1I_1+b_2I_2...$

Linearity theorem:

+ For any electrical circuit containing resistors and independent sources, every node voltage and branch current is a linear function of all source values and has the form of $\sum a_iU_i$

Where: $U_i$, source values; $a_i$, suitably dimensioned constant.

+ Note: the power is not linear function: $P = I^2 R = \dfrac{V^2}{R}$

## 2. SUPERPOSITION

Total current through or voltage across a resistor or branch

+ Determine by adding effects due to each source acting independently

+ Replace a voltage source with a short-circuit

+ Replace a current source with an open-circuit

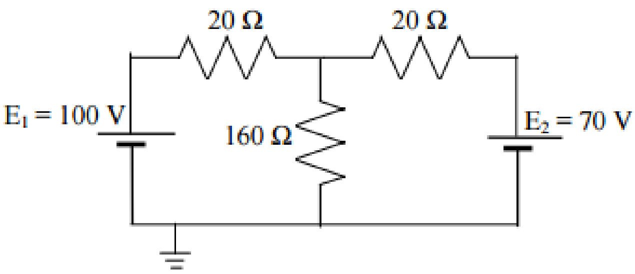
+ Find results of branches using each source independently

+ Algebraically combine results

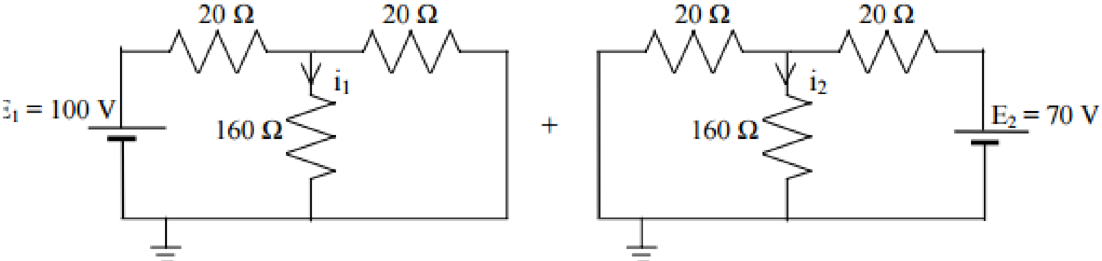
- Power
  - Not a linear quantity
  - Found by squaring voltage or current
- Theorem does not apply to power
  - To find power using superposition
  - Determine voltage or current
  - Calculate power

**Example 1:**

Let us solve the same problem as earlier, but using Superposition theorem.




*Solution*




for circuit 1, source current $= \dfrac{100}{20+160//20} = \dfrac{100}{20+\dfrac{160\times 20}{180}} = \dfrac{100}{37.778} = 2.647\,A$

$\therefore i_1 = 2.647 \times \dfrac{20}{180} = 0.294\text{ A}$

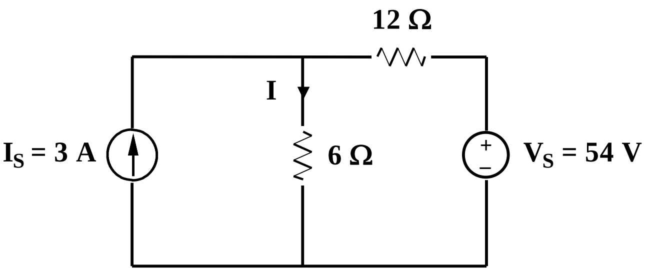
Similarly for circuit 2, source current = $\frac{70}{20+160//20} = \frac{70}{20+\frac{160\times 20}{180}} = \frac{70}{37.778} = 1.853\,A$

$\therefore i_2 = 1.853\times\frac{20}{180} = 0.206$ A

$\therefore$ unknown current $i = i_1 + i_2 = 0.294 + 0.206 = 0.500$ A

which is the same answer that we got from Kirchoff's Laws and Ohm's Law.

**Example 2:** Given the circuit below. Find the current I by using superposition

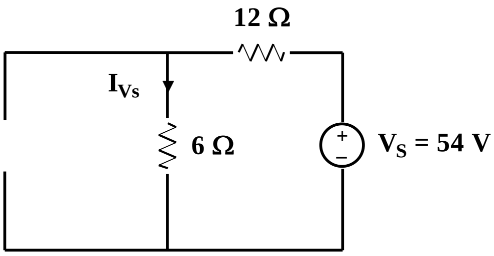



First, deactivate the source $I_S$ and find I in the 6 Ω resistor

Second, deactivate the source $V_S$ and find I in the 6 Ω resistor.

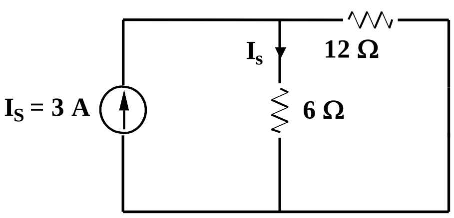
Sum the two currents for the total current.

Step 1: Open-source $I_S$




$I_{Vs} = 3$ A

Step 2: Short-source $V_S$



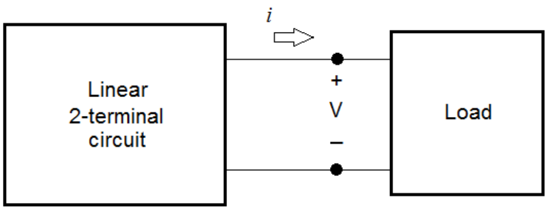

$$I_S = \frac{3x12}{(3+12)} = 2\,A$$

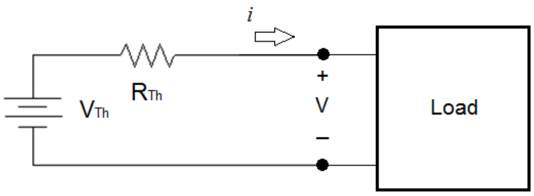
Total current I: $I = I_S + I_{vs} = 5$ A

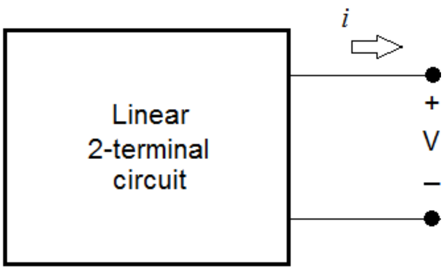
## 3. THEVENIN'S AND NORTON'S THEOREMS

### 3.1. THEVENIN'S THEOREM

- A linear two-terminal circuit can be replaced with an equivalent circuit of an ideal voltage source, $V_{Th}$, in series with a resistor, $R_{Th}$.
- $V_{Th}$ is equal to the open-circuit voltage at the terminals.
- $R_{Th}$ is the equivalent or input resistance when the independent sources in the linear circuit are turned off




**Linear circuit** is a circuit where the voltage is directly proportional to the current (i.e., Ohm's Law is followed).

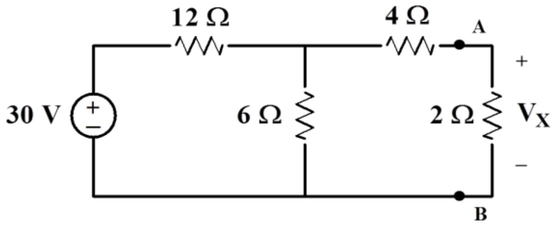
Two terminals are the 2 nodes/2 wires that can make a connection between the circuit to the load.

**Steps to Determine $V_{Th}$ and $R_{Th}$**

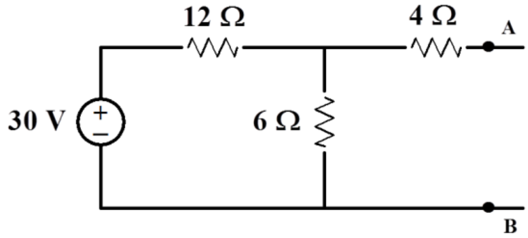
1. Identify the load, which may be a resistor or a part of the circuit.
2. Replace the load with an open circuit.
3. Calculate $V_{OC}$. This is $V_{Th}$.
4. Turn off all independent voltage and currents sources in the linear 2-terminal circuit.
5. Calculate the equivalent resistance of the circuit. This is: $R_{Th}$.
    - The current through and voltage across the load in series with $V_{Th}$ and $R_{Th}$ is the load's actual current and voltage in the original circuit.

**Examples:**

**Ex1.** Find $V_X$ by first finding $V_{TH}$ and $R_{TH}$ to the left of A-B.



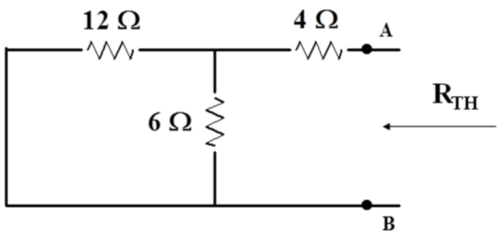

First remove everything to the right of A-B.




$$V_{AB} = 30\,\frac{6\Omega}{6\Omega + 12\Omega} = 10\,V$$

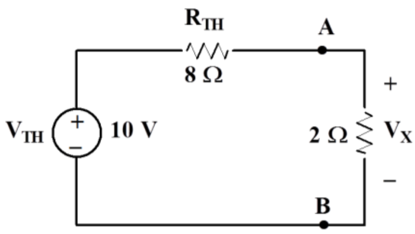
Notice that there is no current flowing in the 4 Ω resistor (A-B) is open. Thus there can be no voltage across the resistor.

We now deactivate the sources to the left of A-B and find the resistance seen looking in these terminals.



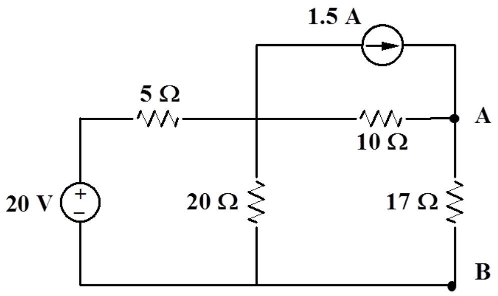

$$R_{TH} = (12\Omega // 6\Omega) + 4\Omega = 8\Omega$$

After having found the Thevenin circuit, we connect this to the load in order to find VX.

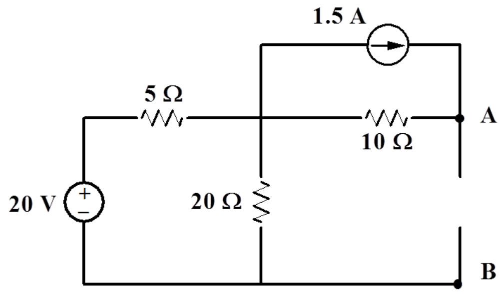



$$V_X = \frac{10V.2\Omega}{2\Omega + 8\Omega} = 2\,V$$

**Ex2.** For the circuit below, find $V_{AB}$ by first finding the Thevenin circuit to the left of terminals A-B.

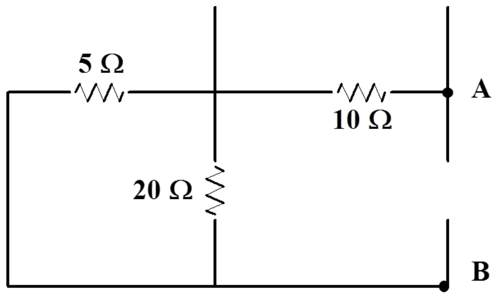



We first find $V_{TH}$ with the 17 Ω resistor removed.

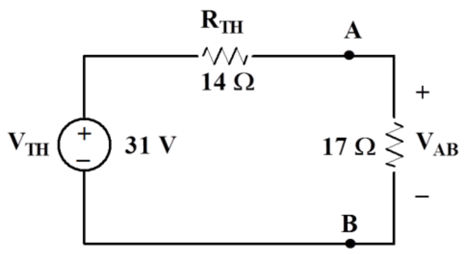



$$V_{AB} = V_{TH} = (1{,}5A).10\Omega + \frac{20V.20\Omega}{20\Omega + 5\Omega} = 31\,V$$

Next, we find $R_{TH}$ by looking into terminals A-B with the sources deactivated.



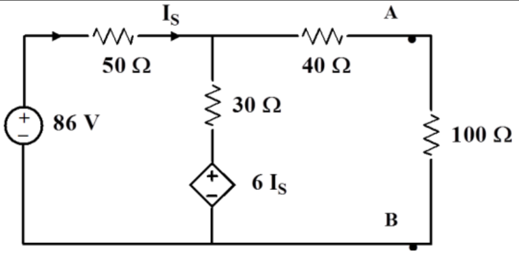

$$R_{TH} = 10 + \frac{5(20)}{(5+20)} = 14\Omega$$



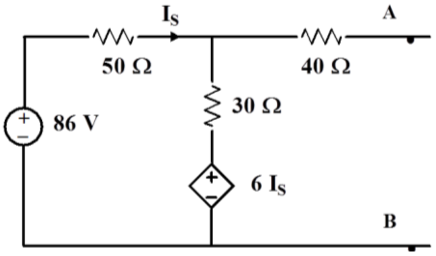

$$V_{AB} = 17V$$

**Working with a mix of independent and dependent sources**

**Ex3.** Find the voltage across the 100 Ω load resistor by first finding the Thevenin circuit to the left of terminals A-B.

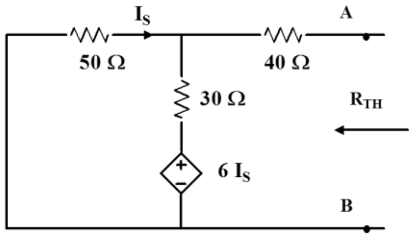



First remove the 100 Ω load resistor and find $V_{AB} = V_{TH}$ to the left of terminals A-B.

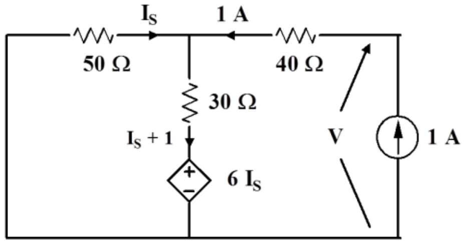



$$-86+80I_S+6I_S=0 \;\rightarrow\; I_S=1A$$
$$V_{AB}=6I_S+30I_S= \;\rightarrow\; 36V$$

To find $R_{TH}$ we deactivate all independent sources but retain all dependent sources as shown in this figure

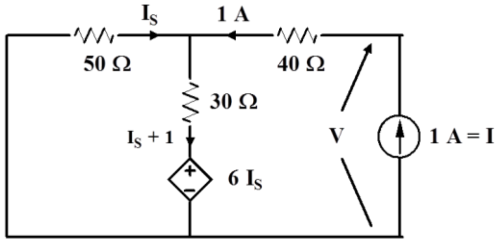



We cannot find $R_{TH}$ of the above circuit, as it stands. We must apply either a voltage or current source at the load and calculate the ratio of this voltage to current to find $R_{TH}$.




Around the loop at the left we write the following equation:

$$50I_S+30(I_S+1)+6I_S=0;\,-->I_S=\frac{-15}{43}A$$

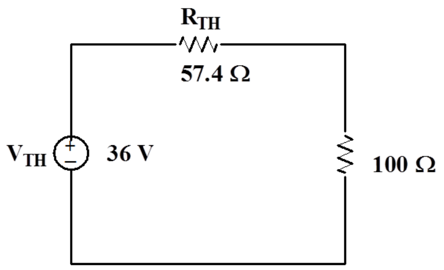



Using the outer loop, going in the cw direction, using drops;

$$50\left(\frac{-15}{43}\right)-1(40)+V=0 \qquad \text{or} \qquad V = 57.4 \text{ volts}$$

$$R_{TH}=\frac{V}{I}=\frac{V}{1}=57.4\ \Omega$$

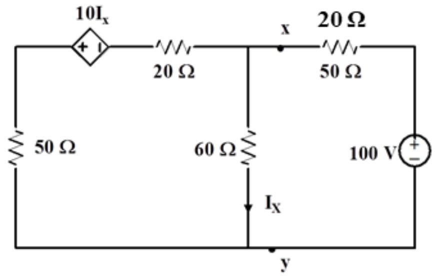
The Thevenin equivalent circuit tied to the 100 Ω load resistor is shown below.




$$V_{100}=\frac{36x100}{57.4+100}=22.9\ V$$

**Ex4.**

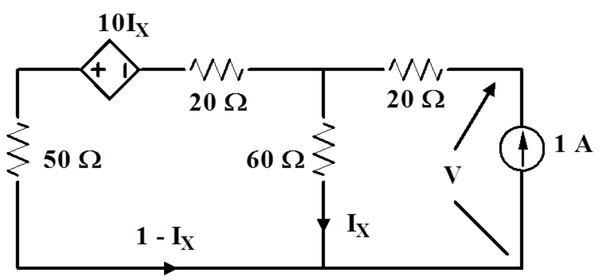
Finding the Thevenin circuit when only resistors and dependent sources are present. Consider the circuit below. Find $V_{xy}$ by first finding the Thevenin circuit to the left of x-y.




For this circuit, it would probably be easier to use mesh or nodal analysis to find $V_{xy}$. However, the purpose is to illustrate Thevenin's theorem.

We first reconcile that the Thevenin voltage for this circuit must be zero. There is no "juice" in the circuit so there cannot be any open circuit voltage except zero. This is always true when the circuit is made up of only dependent sources and resistors.

To find $R_{TH}$ we apply a 1A source and determine V for the circuit below.




Write KVL around the loop at the left, starting at "m", going cw, using drops:

$$-50(1-I_X)+10I_X-20(1-I_X)+60I_X=0$$

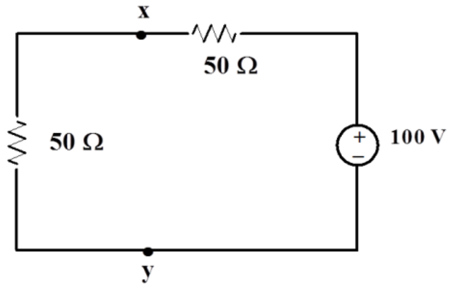
$$I_X=0.5\ A$$

We write KVL for the loop to the right, starting at n, using drops and find;

$$-60(0.5)-1x20+V=0 \qquad V = 50 \text{ volts}$$

$R_{TH} = \frac{V}{I}$; where $V = 50V$; $I = 1A$ ;

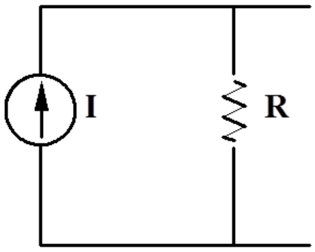
Thus, $R_{TH}$ = 50Ω. The Thevenin circuit tied to the load is given below.



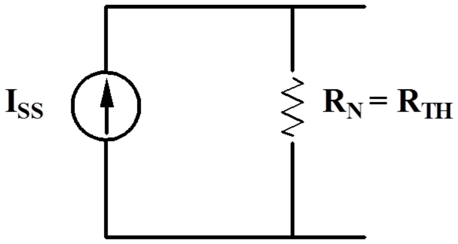

So, $V_{XY}$ = 50 V

### 3.2. NORTON'S THEOREM

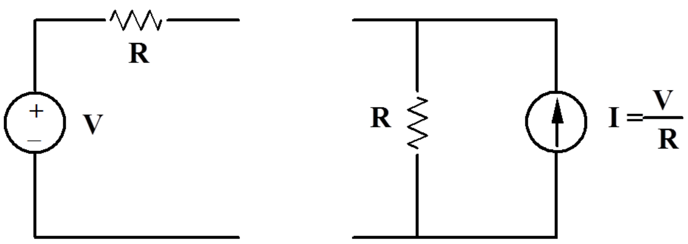
Norton's Theorem states that this network can be replaced by a current source shunted by a resistance R.




In the Norton circuit, the current source is the short circuit current of the network, that is, the current obtained by shorting the output of the network. The resistance is the resistance seen looking into the network with all sources deactivated. This is the same as $R_{TH}$.



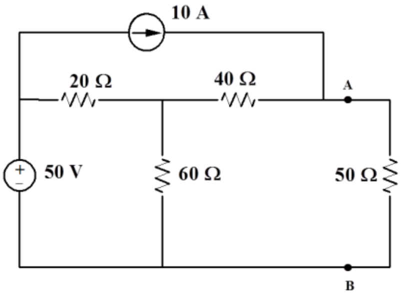

We recall the following from source transformations

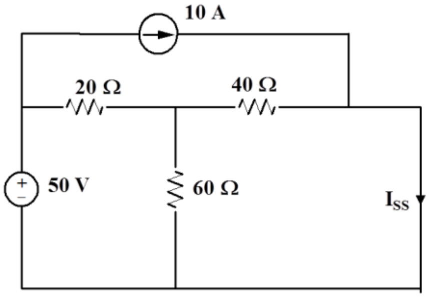


In view of the above, if we have the Thevenin equivalent circuit of a network, we can obtain the Norton equivalent by using source transformation. However, this is not how we normally go about finding the Norton equivalent circuit.

**Ex.**

Find the Norton equivalent circuit to the left of terminals A-B for the network shown below. Connect the Norton equivalent circuit to the load and find the current in the 50 Ω resistor.

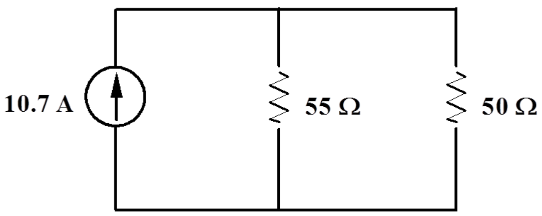



It can be shown by standard circuit analysis that: $I_{SS} = 10.7A$

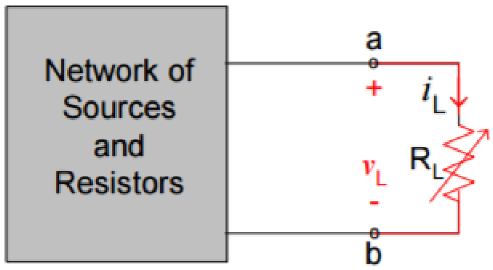
It can also be shown that by deactivating the sources. We find the resistance looking into terminals A-B is: $R_N = 55\Omega$

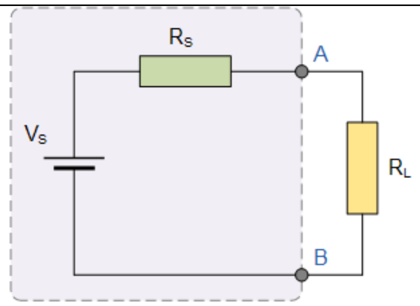
$R_N$ and $R_{TH}$ will always be the same value for a given circuit. The Norton equivalent circuit tied to the load is shown below:




## 4. MAXIMUM POWER TRANSFER THEOREM

The Maximum Power Transfer Theorem is not so much a means of analysis as it is an aid to system design. Simply stated, the maximum amount of power will be dissipated by a load resistance when that load resistance is equal to the Thevenin/Norton resistance of the network supplying the power. If the load resistance is lower or higher than the Thevenin/Norton resistance of the source network, its dissipated power will be less than maximum.




In our Thevenin equivalent circuit above, the maximum power transfer theorem states that "the maximum amount of power will be dissipated in the load resistance if it is equal in value to the Thevenin or Norton source resistance of the network supplying the power".

The process used to make is $R_L = R_{TH} = R_S$ called impedance matching.

**Problem statements:**

- What is the maximum power that can be delivered to a load?
- What is the value of $R_L$ that maximizes the power?
- What is the efficiency of power transfer?

**Power Transfer Calculation:**

$$P_L = \frac{V^2{}_L}{R_L} = \frac{\left(\frac{R_L}{R_L + R_{TH}} V_{TH}\right)^2}{R_L} = \left(\frac{V_{TH}}{R_L + R_{TH}}\right)^2 R_L$$

To find the value of $R_L$ for which $P_L$ is maximum, set: $\frac{dP_L}{dR_L} = 0$;

$$\frac{dP_L}{dR_L} = V_{TH}^2 \left[\frac{(R_{TH} + R_L)^2 - 2R_L(R_{TH} + R_L)}{(R_{TH} + R_L)^4}\right] = 0$$

$$(R_L + R_{TH})^2 = 2R_L(R_L + R_{TH}) \rightarrow R_L = R_{TH}$$

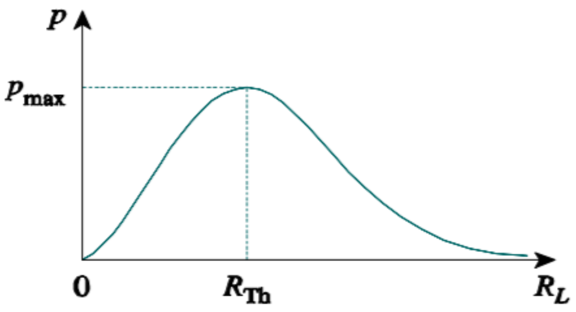
**Results of maximum power transfer:**

- The maximum power transfer takes place when the load resistance match to the Thevenin's resistance
- The maximum power transferred to the load: $P_{max} = P_L, (R_L = R_{TH}) = \frac{V^2{}_{TH}}{4R_{TH}}$
- The efficiency of power transfer:

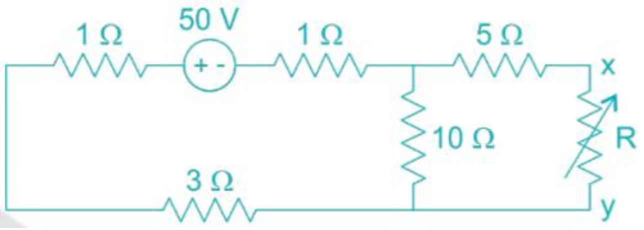
$$\eta = \frac{P_L}{P_{Source}} = \frac{Power\ delivered\ to\ the\ load}{Power\ generated\ by\ the\ source}$$

$$P_{Source} = I^2{}_L\ (R_L + R_{TH}) = \frac{V^2{}_{TH}}{4R_{TH}}(R_L + R_{TH}) = \frac{V^2{}_{TH}}{2R_{TH}}; (when: R_{TH} = R_L)$$

⇨ $\eta = \frac{P_L}{P_{Source}} = 50\%$

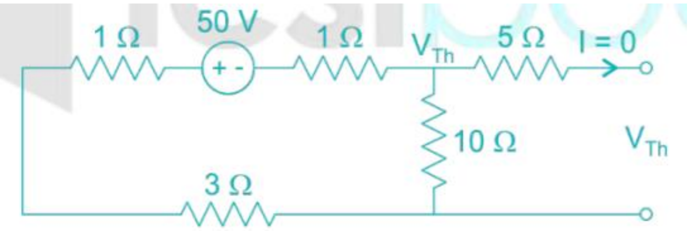



**Example 1:** Find the maximum power delivered to the load R in the given circuit



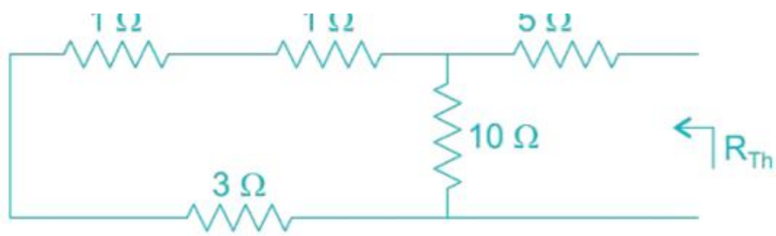

**Solution:**

Step 1: Find $V_{TH}$




$\frac{V_{TH} + 50}{5} + \frac{V_{TH}}{10} + 0 = 0$; ➔ $V_{TH} = -33.3V$

Step 2: Find $R_{TH}$




$R_{TH} = 5\Omega + (10\Omega // 5\Omega) = 8.33\Omega$

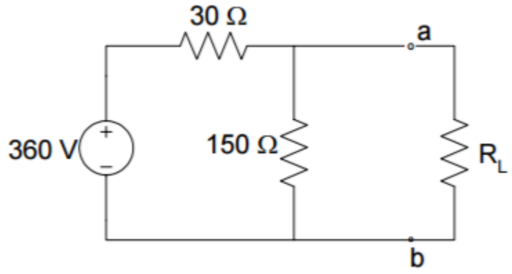
$P_{max} = P_L, (R_L = R_{TH}) = \frac{V^2{}_{TH}}{4R_{TH}} = 33.28W$

**Example 2:**

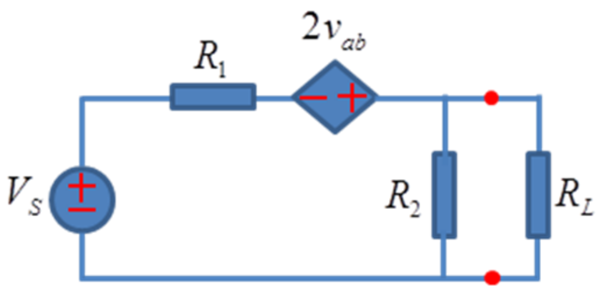
a) Find the value of $R_L$ that result in maximum power being transferred to $R_L$.

b) Calculate the maximum power that can be delivered to $R_L$.

c) When $R_L$ is adjusted for maximum power transfer, what percentage of the power delivered by the 360 V source reaches $R_L$?



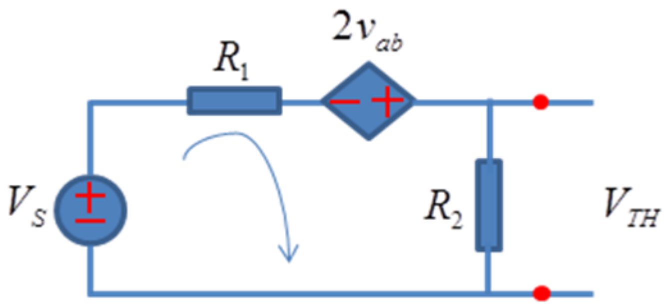

**Example 3:** Find the Load $R_L$ that results in maximum power delivered to the load. Also determine $P_{max}$; $R_1 = 6\Omega$; $R_2 = 4\Omega$; $V_S = 6V$



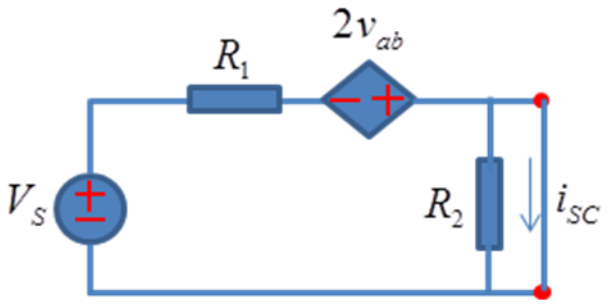

First, we use the circuit (b) to obtain the Thévenin equivalent circuit.

Apply KVL to the close loop:

$6 - 6i + 2v_{ab} - 4i = 0$; and $2v_{ab} = 4i$; $i = 3A$; $V_{TH} = 12V$




Find the short circuit current for circuit (c), $i_{SC}$

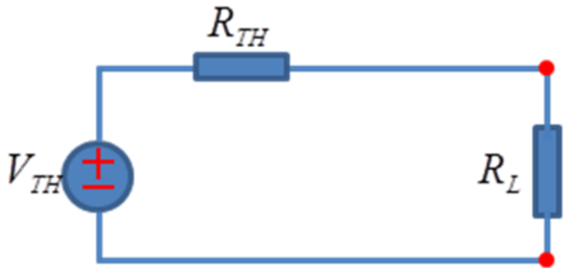



Since ab is short, $v_{ab} = 0$; Apply KVL to the close loop:

$6 - 6i_{SC} + 2v_{ab} = 0$; $i_{SC} = 1A$

Find the equivalent resistance:

$$R_{TH} = \frac{V_{TH}}{i_{SC}} = 12\Omega\ ;\ R_L = R_{TH} = 12\Omega\ ;\ P_{\max} = P_L, (R_L = R_{TH}) = \frac{V^2{}_{TH}}{4R_{TH}} = 3W$$



# CHAPTER 5: FIRST- AND SECOND-ORDER TRANSIENT CIRCUITS

## 1. INTRODUCTION

We perform what is normally referred to as a transient analysis. Beginning with first-order circuits that is, those that contain only a single storage element. When only a single storage element is present in the network, the network can be described by a first-order differential equation.

Our analysis involves an examination and description of the behavior of a circuit as a function of time after a sudden change in the network occurs due to switches opening or closing. Because of the presence of one or more storage elements, the circuit response to a sudden change will go through a transition period prior to settling down to a steady-state value. It is this transition period that we will examine carefully in our transient analysis.

## 2. FIRST-ORDER CIRCUITS

### 2.1. INTRODUCTION

- A first order circuit is characterized by a first order differential equation.
- There are two types of first order circuits:

  + Resistive capacitive, called RC

  + Resistive inductive, called RL
- There are also two ways to excite the circuits:

  + Initial conditions

  + Independent sources

**1. Contents**

+ The Natural Response of an RC Circuit, an RL Circuit

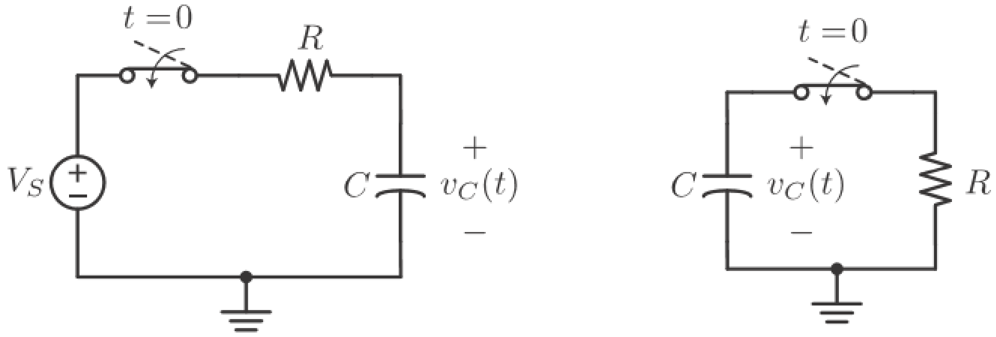
+ The Step Response of RC and RL Circuit

**2. Notion**

The time-varying currents and voltages resulting from the sudden application of sources, usually due to switching, are called transients. By writing circuit equations, we obtain integro-differential equations.

**The Canonical Charging and Discharging RC Circuits**

Consider two different circuits containing both a resistor R and a capacitor C. One circuit also contains a constant voltage source $V_S$; here, the capacitor C is initially uncharged. In the other circuit, there is no voltage source and the capacitor is initially charged to $V_0$.




**The charging and discharging RC circuits**

In both cases, the switch has been open for a long time, and then we flip it at time t = 0. What happens in the circuit throughout the entire experiment? In particular, let's focus on $v_C(t)$, as knowing that will also give us the current $i_C(t)$ by equation:

$$i_C(t) = C\frac{dv_C(t)}{dt}$$

If we follow the same methodology as with resistive circuits, then we'd solve for $v_C(t)$ both before and after the switch closes. Well, before the switch closes, both circuits are in an open state. So $v_C(0^-)$ for the uncharged capacitor is just 0, while it is $V_0$ for the charged capacitor.

After the switch closes, we have complete circuits in both cases. KCL at the node v gives us the two equations for the charging and discharging circuits, respectively:

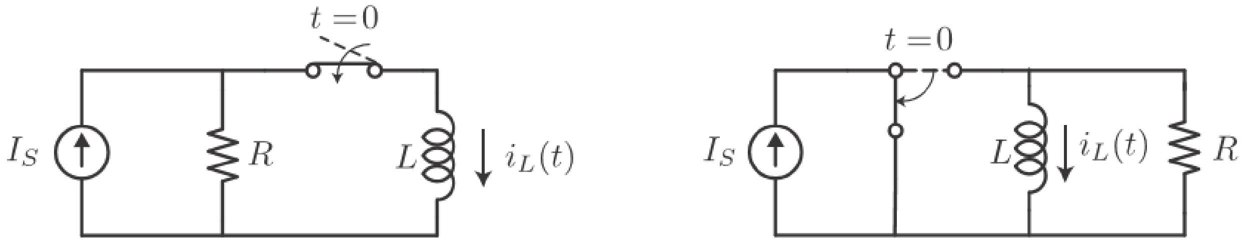
$$v_C(t) + RC\frac{dv_C(t)}{dt} = V_s$$

$$v_C(t) + RC\frac{dv_C(t)}{dt} = 0$$

Notice that we cannot simply solve an algebraic equation and end up with a single value for $v_C(t)$ anymore. Instead, $v_C(t)$ is given by an ordinary differential equation that depends on time. Hence, the function $v_C(t)$ describes the transient response after the switch closes; it is not instantaneous. The other observation you should make is that the equations for both cases are strikingly similar. The task that is now left to us is to solve these ODEs.

**The Canonical Charging and Discharging RL Circuits**

First-order circuits with inductors can be analyzed in much the same way. Consider the "charging" and "dis- charging" RL circuits:




**The "charging" and "discharging" RL circuits**

While the notion of charging an inductor doesn't really make sense, one can think of this in terms of current. In DC steady-state, inductors act as shorts and allow any current to flow through them, but inductors oppose immediate changes in current and introduce delays between the initial and final currents. Again, these time transient responses are given by decaying exponentials. First note that we can derive KVL equations:

$$i_L(t) + \frac{L}{R}\frac{di_L(t)}{dt} = I_S$$

$$i_L(t) + \frac{L}{R}\frac{di_L(t)}{dt} = 0$$

Aside from the time constant, these equations are exactly the same as those for the voltage in a RC circuit. Furthermore, the boundary conditions are analogous; in the charging case, $i_L(0) = 0$ and $i_L(\infty) = I_S$, while for L discharging we have $i_L(0) = I_S$ and $i_L(\infty) = 0$.

The solutions to the ODEs are:

$$i_L(t) = I_S(1 - e^{-t/\tau})$$
$$i_L(t) = I_S e^{-t/\tau}$$

**Circuits with Multiple Resistors, Sources, and Switches**

While the examples that we analyzed were simple and cute, RC and RL circuits can quickly get ugly, as with resistive and amplifier circuits. However, the same techniques that we've used here can be extended to any first-order circuit. Look back at the RC and RL circuit diagrams. The ODEs that we obtained only apply to these circuit configurations, but do they look familiar? Indeed, the capacitor sees a Thevenin circuit, while the inductor sees a Norton circuit! Since we know that any linear circuit has these equivalent circuits, this gives an alternative method for writing down the ODE. Given any circuit with a capacitor or inductor, we can reduce the rest of the circuit down to a Thevenin or Norton equivalent. Thus, for any arbitrary RC or RL circuit with a single capacitor or inductor, the governing ODEs are

$$v_C(t) + R_{Th}C\frac{dv_C(t)}{dt} = v_{Th}(t)$$

$$i_L(t) + \frac{L}{R_N}\frac{di_L(t)}{dt} = i_N(t)$$

where the Thevenin and Norton circuits are those as seen by the capacitor or inductor.

**3. Solution**

**Step 1:**

Choose nodal analysis or mesh analysis approach

The solution of a linear circuit, called dynamic response, can be decomposed into:

**Natural Response + Forced Response**

Or: **Steady Response + Transient Response**

The natural response is due to the initial condition of the storage component (C or L).

The forced response is resulted from external input (or force).

In this chapter, a constant input (DC input) will be considered and the forced response is called step response

**Step 2:**

Differentiate the equation as many times as required to get the standard form of a first order differential equation (D.E)

**Step 3**: Solving the differential equation

$x(t) = x_h(t) + x_p(t)$

(1) homogeneous solution: $x_h(t)$

(2) particular solution: $x_p(t)$

Or $x(t) = x_n(t) + x_f(t)$

(3) natural solution: $x_n(t)$

(4) forced solution: $x_f(t)$

$x_h(t)$ and $x_n(t)$ is due to the initial conditions in the circuit;

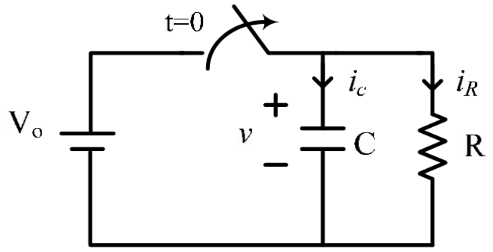
$x_p(t)$ and $x_f(t)$ is due to the forcing functions (independent voltage and current sources for t > 0)

**Step 4:** Find the initial conditions:

$x(0^+)$ then get the unique solution

**2.2. THE NATURAL RESPONSE OF AN RC and RL CIRCUIT**

**A. The source-free series RC circuit**

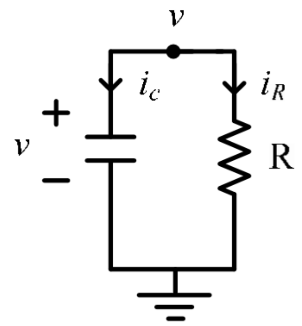



**For:** $t \geq 0$

nodal analysis

$$i_c + i_R = 0$$

$$C\frac{dv}{dt} + \frac{1}{R}v = 0$$




*characteristic root S,*

$$CS + \frac{1}{R} = 0$$

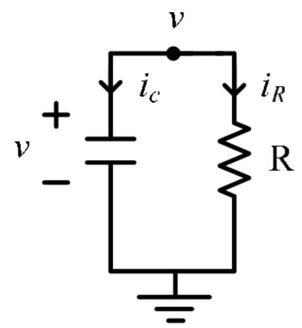
$$S = -\frac{1}{RC}$$

$$\therefore v(t) = Ke^{-\frac{1}{RC}t}, t \geq 0$$

*From the initial condition*

$$v(0^+) = v(0^-) = V_0$$

$$\therefore v(t) = V_0 e^{-\frac{t}{RC}}$$




$t > 0$

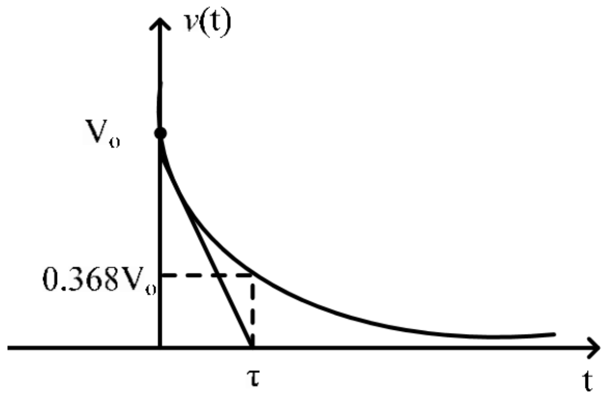
$\tau = RC$ *time constant*

$$v(t) = V_0 e^{-\frac{t}{\tau}}, t \geq 0$$

$$t = \tau \quad , \quad \frac{v(t)}{V_0} = 0.36788$$

$t = 3\tau$ , $4.979\% < 5\%$

$t = 5\tau$ , $0.674\% < 1\%$




It is customary to assume that the capacitor is fully discharged after five time constants $t \geq 5\tau$

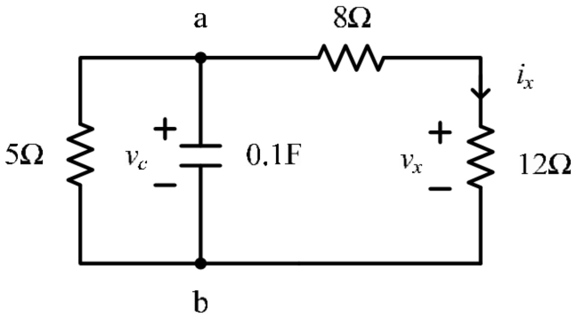
$$i_R(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-\frac{t}{\tau}}, t \geq 0$$

The power dissipated in R is: $p(t) = vi_R = \frac{V_0^{\ 2}}{R} e^{-\frac{2t}{\tau}}$

**Example:**

Find: $v_C(t)$, $v_x(t)$, $i_x(t)$, for $t \geq 0$

$v_C(0) = 15V$




***Solution***

**Step 1**. Use Thevenin theorem to find the equivalent $R_{TH}$ looking into a-b terminals.

$R_{TH} = (8\Omega + 12\Omega)\ parallel\ 5\Omega$ → $R_{TH} = 4\Omega$

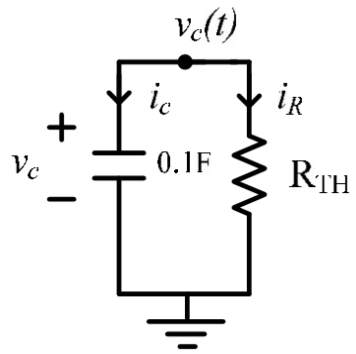
**Step 2**. Find $v_C(t)$

$v_c(0) = 15\text{V}$

$0.1\frac{dv_c}{dt} + \frac{v_c}{4} = 0$

$\boldsymbol{t} = RC = 0.4\ \text{s}$

$v_c(\text{t}) = 15e^{-2.5t}\text{V}, t \geq 0$



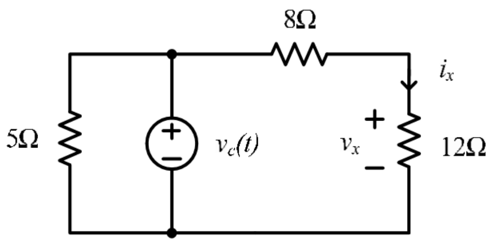

**Step 3**. Replace $v_C(t)$ as a voltage source in the original circuit and solve the resistive circuit.

*By using voltage divider principle*

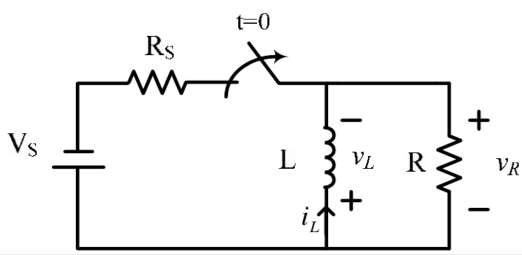
$v_x(t) = \frac{12}{8+12} v_c(t)$

$= 9 \times e^{-2.5t} V, t \geq 0$

$i_x(t) = \frac{v_x(t)}{12} = 0.75e^{-2.5t} A$




**B. The source-free series RL circuit**

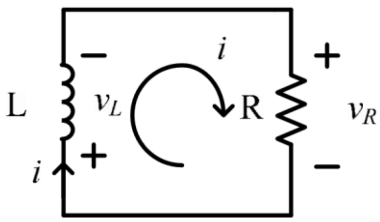


Remember that $v_C(t)$ and $i_L(t)$ are continuous functions for bounded inputs.

$t \le 0$, $i = \frac{V_S}{R_S} = i(0^-) = I_0$

$t > 0$

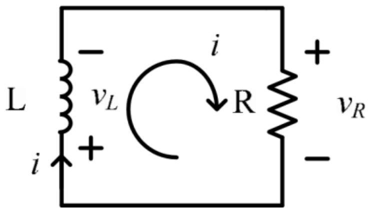
*mesh analysis*




$$v_L + v_R = 0$$

$$L\frac{di}{dt} + Ri = 0$$

*characteristic equation*




$$LS + R = 0$$

$$\therefore S = -\frac{R}{L}$$

$$i(t) = Ke^{-\frac{R}{L}t}, t \ge 0$$

$$i(0^+) = i(0^-) = \mathrm{I}_0$$

$$\therefore i(t) = \mathrm{I}_0 e^{-\frac{t}{\tau}}, t \ge 0 \qquad ; \ \tau = \frac{L}{R}, \text{ time constant}$$

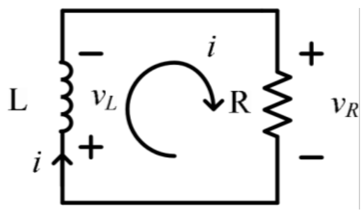
$$v_R(t) = Ri = \mathrm{I}_0 R e^{-\frac{t}{\tau}}$$

L

$$p_R = v_R i = \mathrm{I}_0{}^2 R e^{\frac{-2t}{\tau}}$$

$i$

$$w_R(t) = \int_0^t p_R(t)dt = \frac{1}{2}\mathrm{L}\mathrm{I}_0{}^2(1 - e^{-\frac{2t}{\tau}})$$

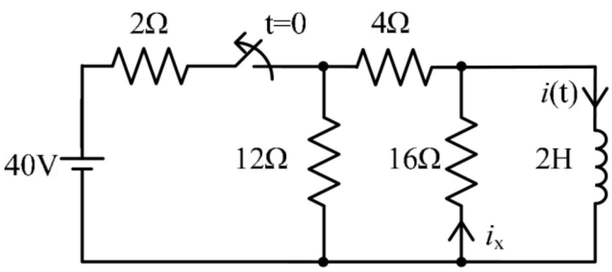



$$w_R(\infty) = \frac{1}{2}LI_0{}^2$$

The switch has been closed for time. At t=0, it is opened.

Find:

$i_x(t)$, $i(t)$, for $t \ge 0$




*Solution:*

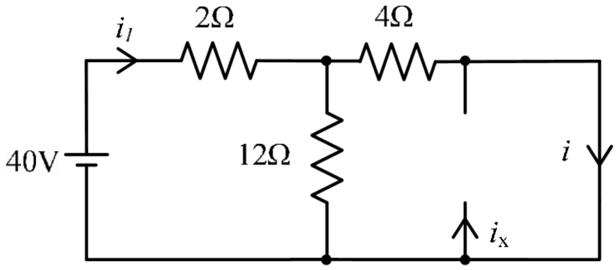
**Step 1:** At, $t < 0$

$$i_1(0^-) = \frac{40\text{V}}{2+(12//4)} = 8\text{ A}$$

$$i_x(0^-) = 0$$

$$i(0^-) = i_1(0^-) \times \frac{12}{12+4} = 8 \times \frac{3}{4} = 6\text{ A}$$

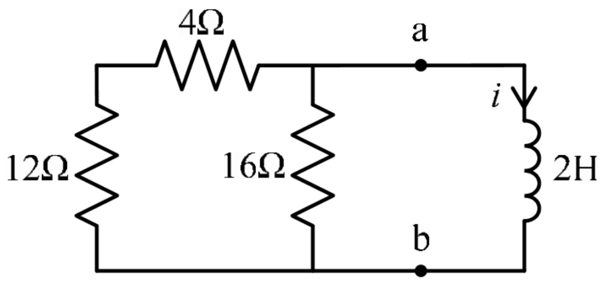
$$\therefore \text{ Initial codition } i(0^+) = i(0^-) = 6\text{ A}$$




**Step 2:** At, $t \geq 0$

$$R_{TH} = (4\Omega + 12\Omega) \text{ } parallel \text{ } 16\Omega \quad \rightarrow$$

$$R_{TH} = 8\Omega$$

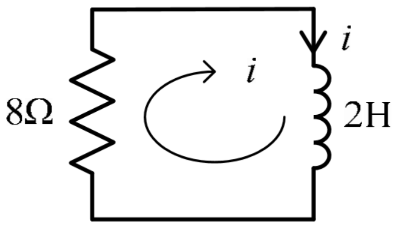



$$2\frac{di}{dt} + 8i = 0$$

$$i(\text{t}) = \text{K e}^{-4\text{t}}, \quad \text{t} \geq 0$$

$$i(0^+) = 6\text{ A}$$

$$\therefore \; i(\text{t}) = 6\text{ e}^{-4\text{t}} \text{ A}, \quad \text{t} \geq 0$$

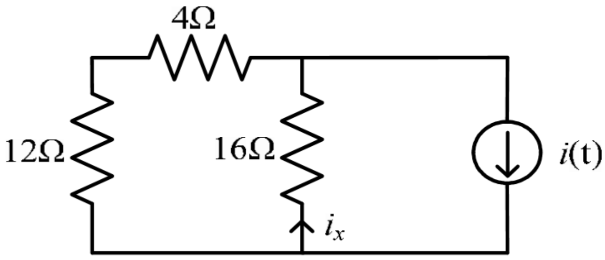



**Step 3**

Replace L with an equivalent current source, and find $i_x(t)$ solve the resistive circuit

$$i_x(\text{t}) = \frac{4+12}{12+4+16} \, i(\text{t})$$

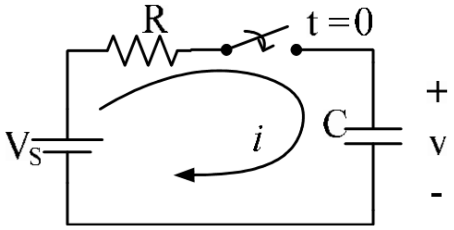
$$= 3\text{e}^{-4\text{t}} \text{ A}, \quad \text{t} \geq 0$$




## 2.3. THE STEP RESPONSE OF RC AND RL CIRCUITS

When a DC voltage (current) source is suddenly applied to a circuit, it can be modeled as a step function, and the resulting response is called step response.

$$V\left(0^-\right)=V_0\ ,\ i=C\frac{dv}{dt}\ ,\ choose\ v\ as\ unknown$$




**Step 1:**

$$RC\frac{dv}{dt}+v=V_S\ ,\ t\geq 0$$

**Step 2**: Solving the differential equation

$(a)$ homogeneous solution

$$RC\frac{dv_h}{dt}+v_h=0$$

$$v_h\left(t\right)=Ke^{-\frac{t}{RC}}\ ,\ t\geq 0$$

$(b)$ particular solution

$$RC\frac{dv_p}{dt}+v_p=V_S$$

$$v_p=V_S$$

(c) complete solution + Initial condition

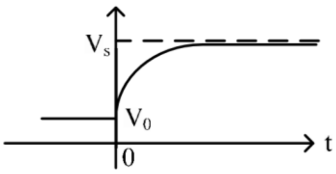
$$\text{v}\left(\text{t}\right)=v_h+v_p$$

$$=Ke^{-\frac{t}{RC}}+V_s$$

$$v\left(0^+\right)=v\left(0^-\right)=V_0$$

$$\therefore\ K=V_0-V_s$$

$$v\left(t\right)=\begin{cases}V_0\ ,\ t<0\\ V_s+\left(V_0-V_s\right)e^{-\frac{t}{\tau}}\ ,\ t\geq 0\end{cases}$$

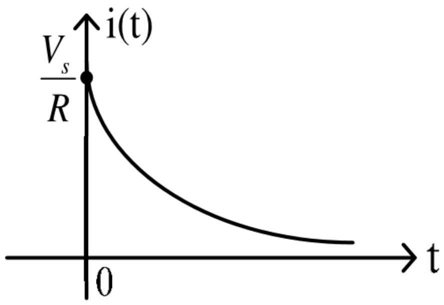



$$i(t)=C\frac{dv}{dt}=C(V_o-V_s)(\frac{-1}{RC})e^{-\frac{t}{RC}}$$

$$=\frac{V_s-V_o}{R}e^{-\frac{t}{RC}}\quad ,\quad t\geq 0$$

$$For\quad V_o=0\quad ,\quad then\quad i(o^+)=\frac{V_S}{R}$$

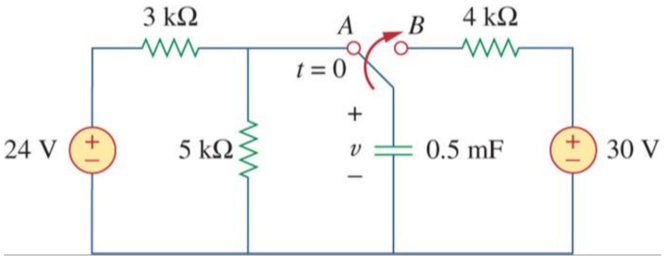
$C$ *is initially short circuited.*



**Example:**

Before t=0, the circuit is under steady state. At t=0, the switch is moved to B.

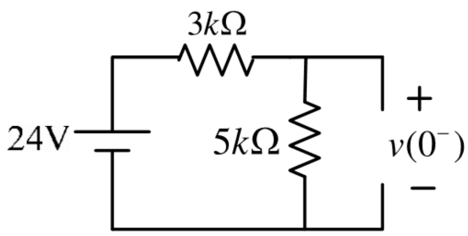
Find: $v(t)$, for $t \ge 0$




*Solution:*

**Step 1:** At, $t < 0$

$$v\left(0^-\right) = \frac{5}{3+5} \times 24\text{V} = 15\text{V}$$




**Step 2:** At, $t > 0$

*mesh analysis*

$$4 \times 10^3 \times \left(0.5 \times 10^{-3}\right)\frac{dv}{dt} + v = 30V$$

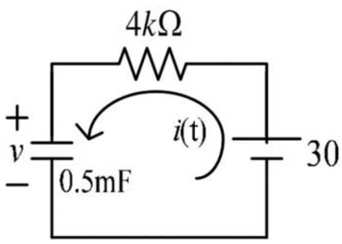
$$v\left(t\right) = v_p + v_h$$

$$= 30 + Ke^{-\frac{t}{\tau}} \quad , \quad t > 0$$

$$\tau = RC = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \ second$$

$$v\left(0^+\right) = v\left(0^-\right) = 15V$$

$$\therefore v\left(t\right) = 30 + \left(15 - 30\right)e^{-\frac{t}{\tau}} \ , t \ge 0$$



**SUMMARY:**

**Procedure for transient analysis of RC and RL circuits.**

1. **Determine the equivalent inductance/capacitance ( $L_{eq}, C_{eq}$ )**

2. **Determine the Thevenin equivalent resistance, $R_{eq}$, seen by ( $L_{eq}, C_{eq}$ )**

3. **The characteristic time is now known $\tau = R_{eq}C_{eq}$ or $\tau = \frac{L_{eq}}{R_{eq}}$**

4. **Calculate the initial value for the voltage/current flowing in the circuit**
   a. **Capacitor acts as an open circuit under dc conditions**
      i. **For a transition happening at $t=0$, $vc(t=0^-) = vc(t=0^+)$**
   b. **Inductor acts as a short circuit under dc conditions**
      i. **For a transition happening at $t=0$, $i_L(t=0^-) = i_L(t=0^+)$**

5. **Estimate the value of $vc$, $i_L$ as $t \to \infty$ (final value)**

6. **The complete solution is:**

$$\text{solution} = \text{final value} + [\text{initial value - final value}]e^{\frac{-t}{\tau}}$$

$$vc(t) = vc_{(t\to\infty)} + \left[vc_{(t=0^+)} - vc_{(t\to\infty)}\right]e^{\frac{-t}{\tau}}$$

$$i_L(t) = i_{L(t\to\infty)} + \left[i_{L(t=0^+)} - i_{L(t\to\infty)}\right]e^{\frac{-t}{\tau}}$$

## 3. SECOND-ORDER CIRCUITS

### 3.1. INTRODUCTION

**1. Contents**

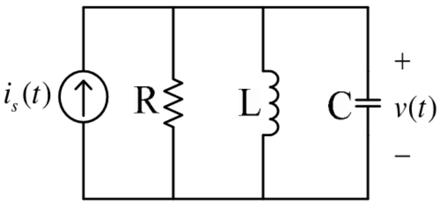
- Linear Second Order Circuits
- Solution Steps
- Finding Initial Values
- The RLC Circuit
- The Natural Response of a Series/Parallel RLC Circuit
- The Step Response of a Series/Parallel RLC Circuit

**2. Notion**

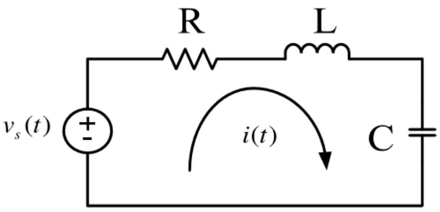
- Circuits containing two energy storage: Inductor and Capacitor
- Described by differential equations that contain second order derivatives

- Need two initial conditions to get the unique solution

**Example:**




**(a) RLC parallel circuit**




**(b) RLC series circuit**

**3. Solution steps**

**Step1**: Choose nodal analysis or mesh analysis approach

**Step2**: Differentiate the equation as many times as required to get the standard form of a second order differential equation (D.E)

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+x=y(t)$$

**Step 3**: Solving the differential equation

$$x(t)=x_h(t)+x_p(t)$$

(5) homogeneous solution: $x_h(t)$

(6) particular solution: $x_p(t)$

Or $x(t)=x_n(t)+x_f(t)$

(7) natural solution: $x_n(t)$

(8) forced solution: $x_f(t)$

$x_h(t)$ and $x_n(t)$ is due to the initial conditions in the circuit;

$x_p(t)$ and $x_f(t)$ is due to the forcing functions (independent voltage and current sources for t > 0)

**Step 4:** Find the initial conditions:

$x(0^+)$, $\frac{dx(0^+)}{dt}$; and then get the unique solution

**NOTE:**

a. **The forced response**

The forced response is due to the independent sources in the circuit for t > 0. Since the natural response will die out once the circuit reaches steady-state (under DC conditions), the forced response can be found by analyzing the circuit at t = ∞: $x_f(t)=x(\infty)$

**b. The natural response**

A 2nd -order differential equation has the form:

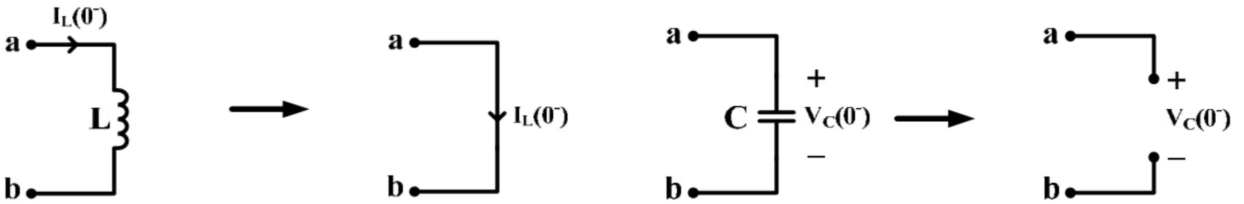
$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+x=y(t)$$ ; Where $x(t)$ is a voltage $v(t)$ or a current $i(t)$.

To find the natural response, set the forcing function $y(t)$ (the right-hand side of the D.E) to zero.

$$a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+x=0$$

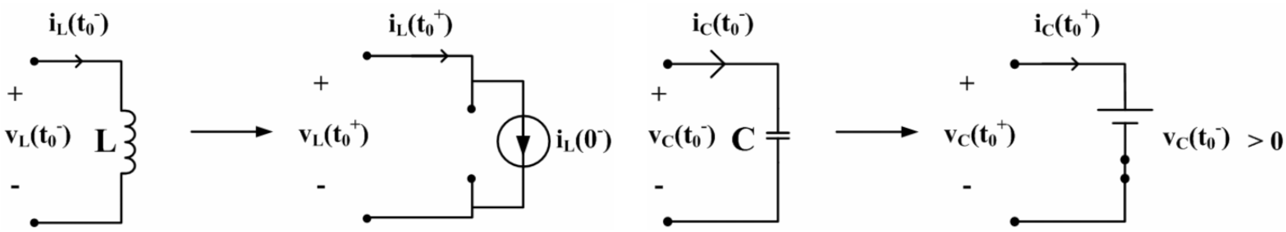
**4. Finding initial values**

- Under **DC steady state**, L is like a short circuit and C is like an open circuit.




**(a) L is like a short circuit** **(b) C is like an open circuit**

- Under **transient condition**, L is like an open circuit and C is like a short circuit because $i_L(t)$ and $v_C(t)$ are continuous functions if the input is bounded.




**(a) L is like an open circuit** **(b) C is like a short circuit**

- To find $\frac{di_L(0^+)}{dt}$ and $\frac{dv_C(0^+)}{dt}$, we use the relations as bellow:

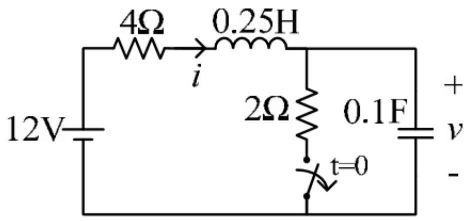
$$L\frac{di_L(0^+)}{dt}=v_L(0^+); \text{ and } C\frac{dv_C(0^+)}{dt}=i_C(0^+)$$

One can find $v_L(0^+)$, $i_C(0^+)$ using either nodal or mesh analysis.

# EXAMPLES

**Example 1.**

The circuit is under steady state. The switch is opened at t = 0, determine the initial values:

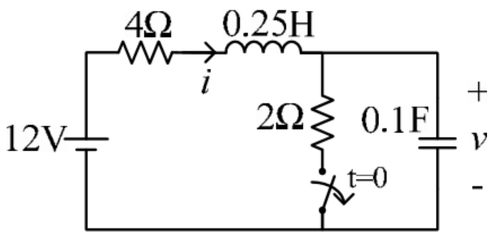


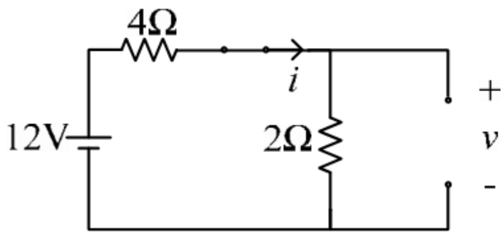

$i(0^+)$, $\dfrac{di\ (0^+)}{dt}$, $v(0^+)$, $\dfrac{dv\ (0^+)}{dt}$

$i(\infty)$, $v(\infty)$

*Solution:*

- $t < 0$, the circuit is under DC steady state. L is like a short circuit and C is like an open circuit.



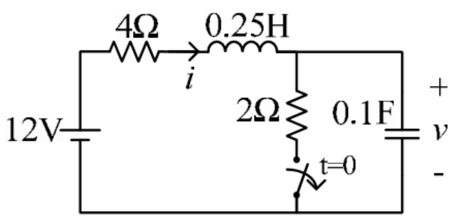



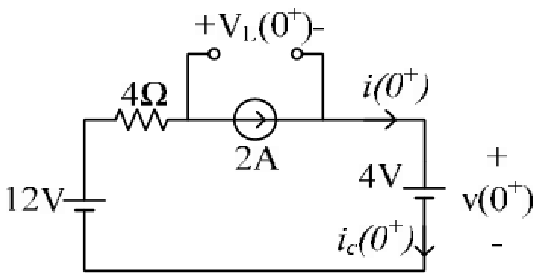

$i(0^-) = 2A$; $v(0^-) = 4V$, we have: $t < 0$

$i(0^+) = i(0^-) = 2A$ and $v(0^+) = v(0^-) = 4V$

- $t > 0$, the switch is opened at t = 0. The circuit is under **transient condition**, L is like an open circuit and C is like a short circuit

(Since the inductor cannot current change abruptly. The inductor can be treated as a current source in this case; Since the capacitor voltage cannot change abruptly. The capacitor can be treated as a voltage source in this case).




$t > 0$

KVL:

$$2A.4 + v_L(0^+) + 4V = 12V\ ;\ \rightarrow\ v_L(0^+) = 0V\ ;\ \frac{di\ (0^+)}{dt} = \frac{v_L(0^+)}{L} = 0$$

KCL:

$$i_C(0^+)=i_C(0^-)=2A;\ \frac{dv_C(0^+)}{dt}=\frac{i_C(0^+)}{C}=20\,V/S$$

- $t=\infty$; L is short circuit, C is open

$i\ (\infty)=0A;\ v\ (\infty)=12V$

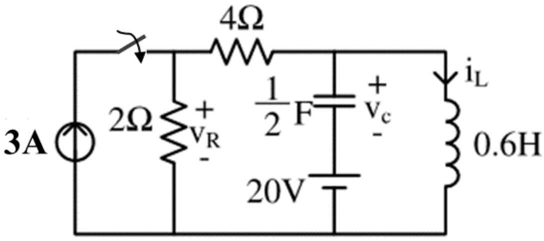
**<u>Example 2</u>**

Find:

$i_L(0^+)$, $\frac{di_L(0^+)}{dt}$, $v_C(0^+)$, $\frac{dv_C(0^+)}{dt}$

$i_L(\infty)$, $v_C(\infty)$, $v_R(0^+)$, $v_R(\infty)$,

$\frac{dv_C(0^+)}{dt}$

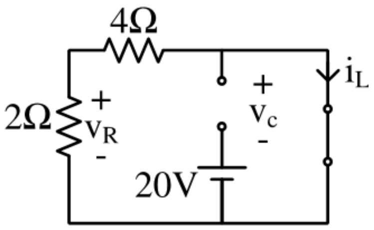



*<u>Solution:</u>*

- $t<0$, the circuit is under DC steady state. L is like a short circuit and C is like an open circuit.

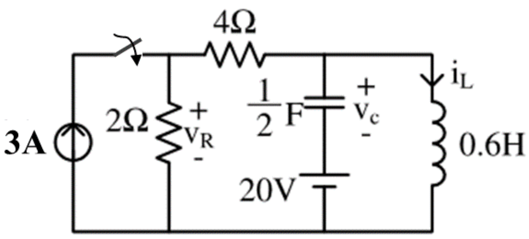
$i_L(0^-)=0A$ , $v_C(0^-)=-20V$ ,

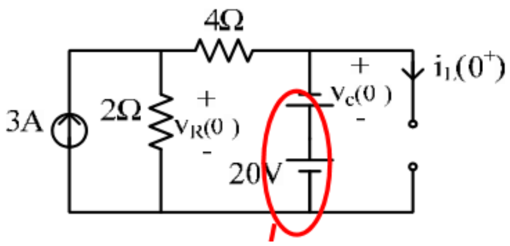
$v_R(0^-)=0V$

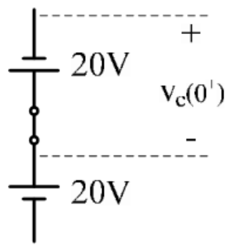



$t>0$

- $t=0^+$, the switch is closed at t = 0. The circuit is under transient condition, L is like an open circuit and C is like a short circuit

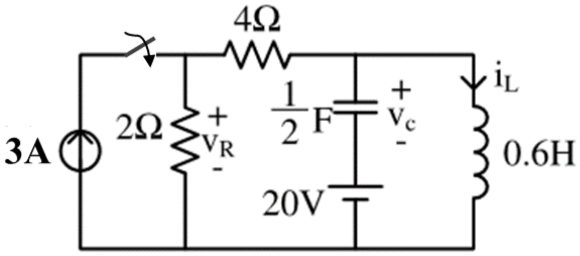


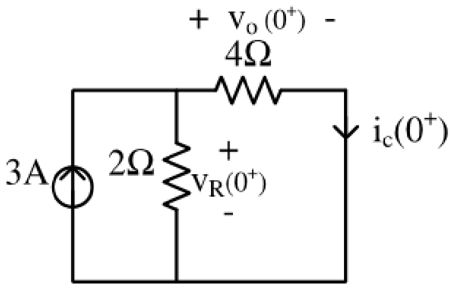




$i_L(0^+)=i_L(0^-)=0A$

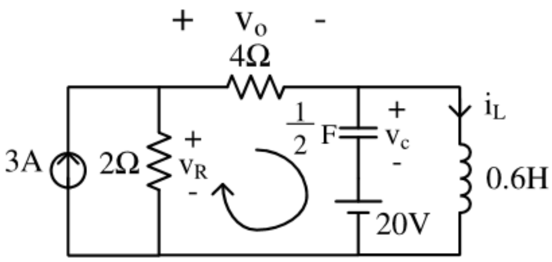
$v_C(0^+)=v_C(0^-)=-20V$




$$i_{2\Omega}(0^+) = 3A.\frac{4\Omega}{2\Omega + 4\Omega} = 2A;\ v_R(0^+) = 2A.2\Omega = 4V;\ i_C(0^+) = 3A.\frac{2\Omega}{2\Omega + 4\Omega} = 1A$$

$$\frac{di_L(0^+)}{dt} = \frac{v_L(0^+)}{L} = 0A/S;\ \frac{dv_C(0^+)}{dt} = \frac{i_C(0^+)}{C} = 2V/S$$




From KVL:

$$-v_R + v_0 + v_C + 20 = 0$$

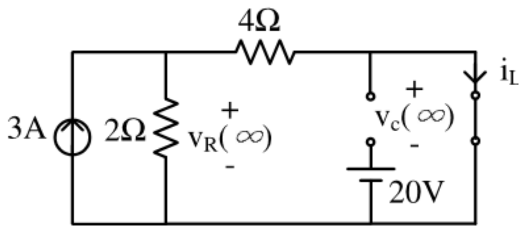
Take derivative: $\frac{dv_R(t)}{dt} = \frac{dv_0(t)}{dt} + \frac{dv_C(t)}{dt}$; → $\frac{dv_R(0^+)}{dt} = \frac{dv_0(0^+)}{dt} + \frac{dv_C(0^+)}{dt}$ (eq – 1)

From KCL:

$3A = \frac{v_R(t)}{2\Omega} + \frac{v_0(t)}{4\Omega}$; → $0 = \frac{dv_R(t)}{2dt} + \frac{dv_0(t)}{4dt}$ (eq – 2)

Combining (eq – 1) and (eq – 2); $\frac{dv_R(0^+)}{dt} = \frac{2}{3}V/S$

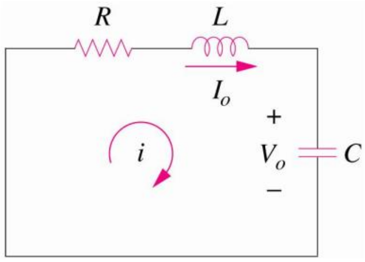
- $t = \infty$; L is short circuit, C is open




$$i_L(\infty) = 3A.\frac{2\Omega}{2\Omega + 4\Omega} = 1A;\ v_C(\infty) = -20V;\ v_R(\infty) = 3A.\frac{2\Omega.4\Omega}{2\Omega + 4\Omega} = 4V$$

## 3.2. NATURAL RESPONSE OF A SERIES/PARALLEL RLC CIRCUIT

### A. The source-free series RLC circuit




Assumed initial conditions :

$i(0) = I_0$

$v_C(0) = \frac{1}{C}\int_{-\infty}^{0} idt = V_0$

**+ Step 1:**

Mesh analysis: $Ri + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} idt = 0$ ;

To eliminate the integral, we take derivative: $L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0 \rightarrow \frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$

**+ Step 2:**

Homogeneous solution, characteristic equation

$S^2 + 2\alpha S + \omega_0^2 = 0$; where $\alpha = \frac{R}{2L}$; Damping factor; and $\omega_0^2 = \frac{1}{LC}$; $\omega_0$, Resonant frequency (or undamped natural frequency)

Characteristic roots (natural frequencies):

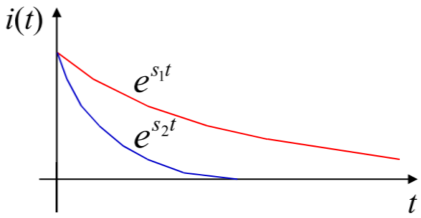
$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$ and $S_1 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$ ; $i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$

Where $A_1$ and $A_2$ are determined from the initial conditions: $i(0^+)$ ; $\frac{di(0^+)}{dt}$

<u>Three cases discussed</u>

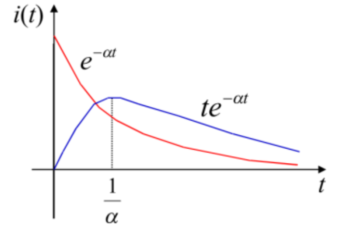
**Overdamped case (distinct real roots):** $\alpha > \omega_0$

Two real roots; $i(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t}$




**Critically damped case (repeated real root):** $\alpha = \omega_0$

Equal real roots; $i(t) = (A_1 + A_2.t)e^{-\alpha t}$ , because: $S_1 = S_2 = -\frac{R}{2L} = -\alpha$

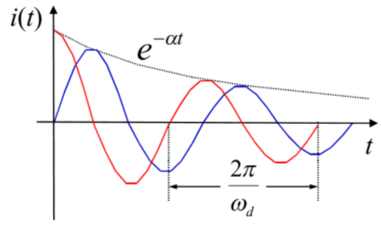



**Underdamped case (complex-conjugate roots):** $\alpha < \omega_0$

Complex conjugate roots:

$$S_1 = -\alpha + j\omega_d\,;\; S_2 = -\alpha - j\omega_d\,;\; \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

$$i(t) = A_1 e^{-\alpha t}\cos(\omega_d t) + A_2 e^{-\alpha t}\sin(\omega_d t)$$




**Note:** once $i(t)$ is obtained, solutions of other variables can be obtained from this mesh current.

**+ Step 3:**

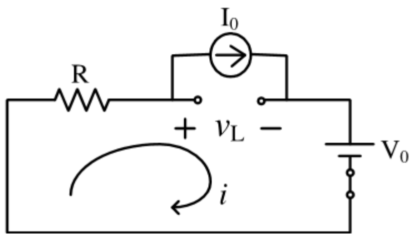
Initial Condition: $t = 0^+; i(0^+) = i(0^-) = I_0$; from mesh equation, let $t = 0^+$;

$$Ri(0^+) + L\frac{di(0^+)}{dt} + \frac{1}{C}\int_{-\infty}^{0^+} idt = 0\,; \rightarrow \frac{di(0^+)}{dt} = -\frac{R}{L}i(0^+) - \frac{V_0}{L} = -\frac{R}{L}I_0 - \frac{V_0}{L}$$

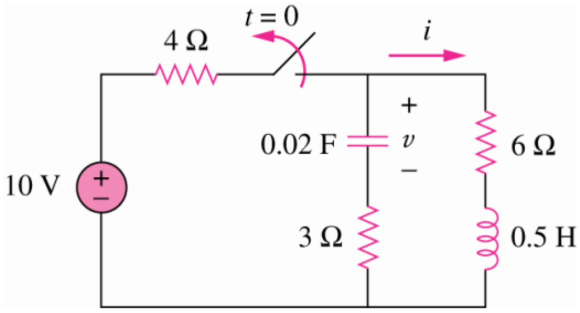
Or from equivalent circuit at $t = 0^+$

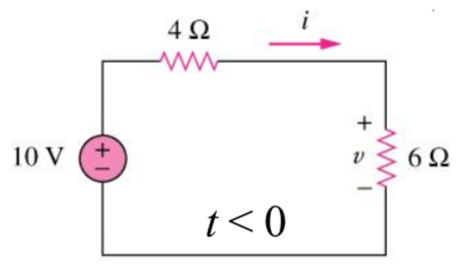
$$L\frac{di(0^+)}{dt} = v_L = -(I_0 R + V_0)$$

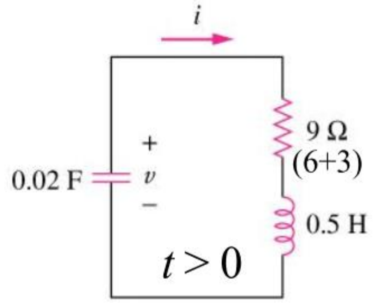
$$\frac{di(0^+)}{dt} = -\frac{(I_0 R + V_0)}{L}$$




<u>**Example 1**</u>: find $i(t)$






- $t < 0$, the circuit is under DC steady state. L is like a short circuit and C is like an open circuit.

$$i(0^-) = \frac{10V}{4\Omega + 6\Omega} = 1A;\ v(0^-) = i(0^-).6\Omega = 6V;$$

- $t = 0^+$; $\alpha = \frac{R}{2L} = \frac{9\Omega}{2*0.5H} = 9$; $\omega_0^2 = \frac{1}{LC} = \frac{1}{0.5*0.02} = 100$; → $\alpha < \omega_0$, Underdamped case (complex-conjugate roots); $\omega_d = \sqrt{\omega_0^2 - \alpha^2} = 4.359$

$$\frac{d^2i}{dt^2} + \frac{9}{0.5}\frac{di}{dt} + \frac{1}{0.5*0.02}i = 0;$$

Characteristic equation:

$$S^2 + 18S + 100 = 0;\ S_1 = -\alpha + j\omega_d = -9 + j4.359;\ S_2 = -\alpha - j\omega_d = -9 - j4.359$$

$$i(t) = A_1e^{-\alpha t}\cos(\omega_d t) + A_2e^{-\alpha t}\sin(\omega_d t);\ i(t) = A_1e^{-9t}\cos(4.359t) + A_2e^{-9t}\sin(4.359t)$$

- Initial conditions:

$$i(0^+) = i(0^-) = 1A;\ v_C(0^+) = v(0^-) = 6V$$

From mesh equation, let $t = 0^+$;

$$Ri(0^+) + L\frac{di(0^+)}{dt} + \frac{1}{C}\int_{-\infty}^{0^+} idt = 0; \rightarrow \frac{di(0^+)}{dt} = -\frac{R}{L}i(0^+) - \frac{v_C(0)^+}{L} = -2(9*1-6) = -6\ A/S$$

We have:

$$i(t) = A_1e^{-9t}\cos(4.359t) + A_2e^{-9t}\sin(4.359t) \rightarrow$$

$$\frac{di(t)}{dt} = -4.359\,A_1e^{-9t}\sin(4.359t) - 9A_1e^{-9t}\cos(4.359t) + 4.359\,A_2e^{-9t}\cos(4.359t) - 9A_1e^{-9t}\sin(4.359t)$$

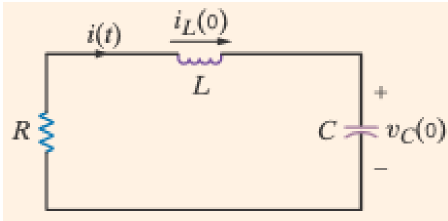
$$i(0^+) = i(0^-) = 1A = A_1;\ \frac{di(0^+)}{dt} = -9A_1 + 4.359A_2 = -6 \rightarrow A_2 = 0.6882$$

$$\rightarrow i(t) = e^{-9t}\cos(4.359t) + 0.6882e^{-9t}\sin(4.359t)$$

**Example 2** (example 7.8, pp. 325-326)

C=0.04F, L=1H, R=6Ω, $i_L(0)$=4A, and $v_C(0)$= -4V

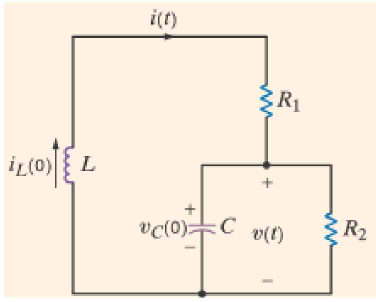
Find $i(t)$




**Example 3** (example 7.9, pp. 327-326)

C=1/8 F, L=2H, $R_1$=10Ω, $R_2$=8Ω $i_L(0)$= 0.5 A, and $v_C(0)$=1 V

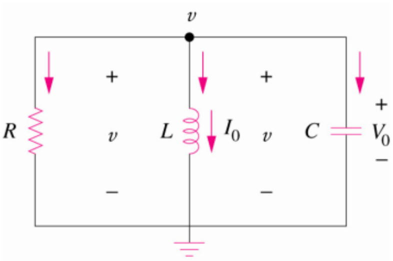
Find $v(t)$




**B. The source-free parallel RLC Circuit**

Assumed initial conditions:

$$\begin{cases} i(0) = I_0 = \dfrac{1}{L}\displaystyle\int_{\infty}^{0} v(t)dt & (1a) \\ v(0) = V_0 & (1b) \end{cases}$$




Applying KCL gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v dt + C\frac{dv}{dt} = 0 \quad (2)$$

$$\Rightarrow \frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0 \quad (3)$$

Let $v(t) = Ae^{st}$, the characteristic equation becomes

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where $\begin{cases} \alpha = \dfrac{1}{2RC} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$

**Summary**

- Overdamped case : $\alpha > \omega_0$

  $v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- Critically damped case : $\alpha = \omega_0$

  $v(t) = (A_1 + A_2 t)e^{-\alpha t}$

- Underdamped case : $\alpha < \omega_0$

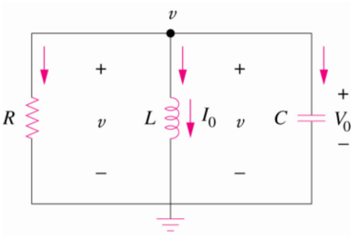
  $s_{1,2} = -\alpha \pm j\omega_d$

  where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

  $v(t) = e^{-\alpha t}(A_1 \cos\omega_d t + A_2 \sin\omega_d t)$

Finding the constants: $A_1$ and $A_2$

To determine $A_1$ and $A_2$,
we need $v(0)$ and $dv(0)/dt$.




1. $\boxed{v(0) = V_0}$
2. KCL at $t = 0$ gives

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

or $$\boxed{\frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}}$$

**NOTE:**

- Series *RLC* Circuit

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\text{where } \begin{cases} \alpha = \dfrac{R}{2L} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$

Initial conditions:

$$\begin{cases} i(0) = I_0 \\ \dfrac{di(0)}{dt} = -\dfrac{(V_0 + RI_0)}{L} \end{cases}$$

- Parallel *RLC* Circuit

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

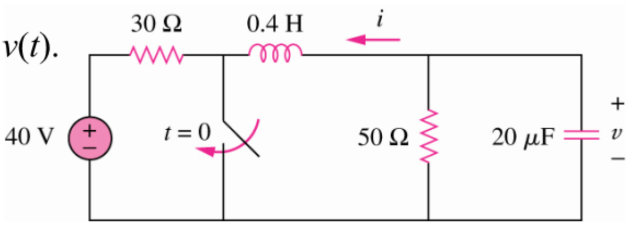
$$\text{where } \begin{cases} \alpha = \dfrac{1}{2RC} \\ \omega_0 = \dfrac{1}{\sqrt{LC}} \end{cases}$$

Initial conditions:

$$\begin{cases} v(0) = V_0 \\ \dfrac{dv(0)}{dt} = -\dfrac{(V_0 + RI_0)}{RC} \end{cases}$$

**Example**:

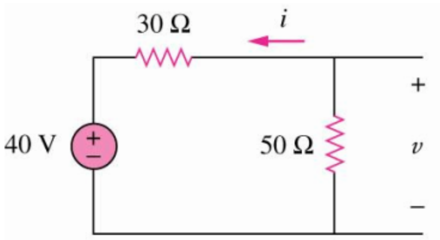
Find $v(t)$.




*Solution*

$t < 0$, the circuit is under DC steady state. L is like a short circuit and C is like an open circuit.

→Get $x(0^-)$: $i(0^-)$, $v(0^-)$

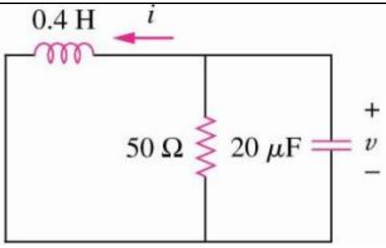



$t < 0$

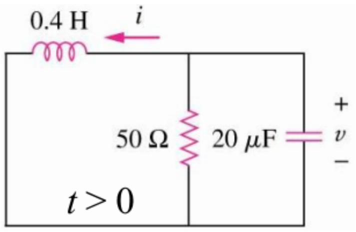
$t > 0$,

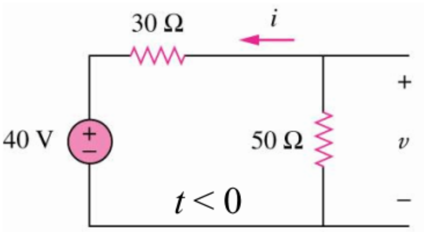
→Get $x(\infty)$: $i(\infty)$, $v(\infty)$

And $\frac{dx(0)}{dt}$; $S_{1,2}$; $A_{1,2}$




$t > 0$




$$\begin{cases} \alpha = \frac{1}{2RC} = 500 \\ \omega_0 = \frac{1}{\sqrt{LC}} = 354 \end{cases}$$

$$\Rightarrow s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -854, -146$$

$$v(t) = A_1 e^{-854t} + A_2 e^{-146t}$$

From the initial conditions :

$$\begin{cases} v(0) = \frac{50}{30+50}(40) = 25\ V \\ i(0) = -\frac{40}{30+50} = -0.5\ A \\ \frac{dv(0)}{dt} = -\frac{v(0)+Ri(0)}{RC} = \frac{25-50\times 0.5}{50\times 20\times 10^{-6}} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A_1 = -5.156 \\ A_2 = 30.16 \end{cases}$$

### 3.3. STEP RESPONSE OF A SERIES/PARALLEL RLC CIRCUIT

#### A. The step response of a series RLC circuit

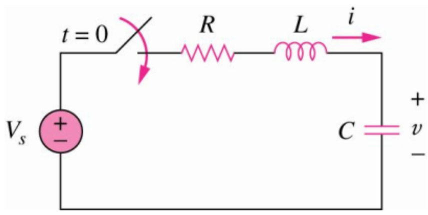
Applying KVL for $t > 0$,

$$Ri + L\frac{di}{dt} + v = V_S \quad (1)$$

But $i = C\frac{dv}{dt}$

$$\Rightarrow \boxed{\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_S}{LC}} \quad (2)$$

(2) has the same form as in the source-free case.




$v(t) = v_t(t) + v_{ss}(t)$

where

$$\begin{cases} v_t : \text{the transient response} \\ v_{ss} : \text{the steady-state response} \end{cases}$$

**Characteristic Equation:**

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v-V_S}{LC}=0$$

Let $v^{'}=v-V_S$,

$$\Rightarrow \frac{d^2v^{'}}{dt^2}+\frac{R}{L}\frac{dv^{'}}{dt}+\frac{v^{'}}{LC}=0$$

The characteristic equation becomes

$$\Rightarrow s^2+\frac{R}{L}s+\frac{1}{LC}=0$$

Same as in the source - free case.

**Summary:**

$$\Rightarrow v(t)=v_t(t)+v_{ss}(t)$$

where $\begin{cases} v_t(\infty)=0 \\ v_{ss}(\infty)=v(\infty)=V_S \end{cases}$

$$v_t(t)=\begin{cases} A_1e^{s_1t}+A_2e^{s_2t} & \text{(Overdamped)} \\ (A_1+A_2t)e^{-\alpha t} & \text{(Critically damped)} \\ (A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t} & \text{(Underdamped)} \end{cases}$$

where $A_{1,2}$ are obtained from $v(0)$ and $dv(0)/dt$.

**B. The step response of a parallel RLC circuit**

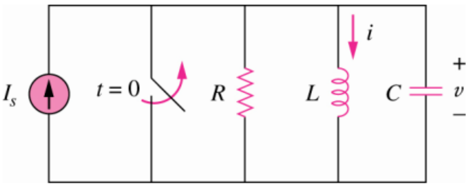
Applying KCL for $t>0$,

$$\frac{v}{R}+i+C\frac{dv}{dt}=I_S \qquad (1)$$

But $v=L\dfrac{di}{dt}$

$$\Rightarrow \frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i}{LC}=\frac{I_S}{LC} \qquad (2)$$

(2) has the same form as in the source - free case.




$i(t)=i_t(t)+i_{ss}(t)$

where

$\begin{cases} i_t : \text{the transient response} \\ i_{ss} : \text{the steady - state response} \end{cases}$

**Characteristic Equation:**

$$\frac{d^2i}{dt^2}+\frac{1}{RC}\frac{di}{dt}+\frac{i-I_S}{LC}=0$$

Let $i^{'}=i-I_S$ ,

$$\Rightarrow \frac{d^2i^{'}}{dt^2}+\frac{1}{RC}\frac{di^{'}}{dt}+\frac{i^{'}}{LC}=0$$

The characteristic equation becomes

$$\Rightarrow s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

Same as in the source - free case.

**Summary:**

$$\Rightarrow i(t)=i_t(t)+i_{ss}(t)$$

$$\text{where } \begin{cases} i_t(\infty)=0 \\ i_{ss}(t)=i(\infty)=I_S \end{cases}$$

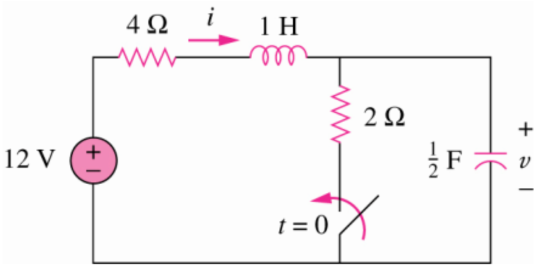
$$i_t(t)=\begin{cases} A_1e^{s_1t}+A_2e^{s_2t} & \text{(Overdamped)} \\ (A_1+A_2t)e^{-\alpha t} & \text{(Critically damped)} \\ (A_1\cos\omega_d t+A_2\sin\omega_d t)e^{-\alpha t} & \text{(Underdamped)} \end{cases}$$

where $A_{1,2}$ are obtained from $i(0)$ and $di(0)/dt$.

**NOTE: General Second-Order Circuits**

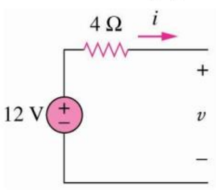
- Steps required to determine the step response.
  - Determine $x(0)$, $dx(0)/dt$, and $x(\infty)$.
  - Find the transient response $x_t(t)$.
    - Apply KCL and KVL to obtain the differential equation.
    - Determine the characteristic roots ($s_{1,2}$).
    - Obtain $x_t(t)$ with two unknown constants ($A_{1,2}$).
  - Obtain the steady-state response $x_{ss}(t) = x(\infty)$.
  - Use $x(t) = x_t(t) + x_{ss}(t)$ to determine $A_{1,2}$ from the two initial conditions $x(0)$ and $dx(0)/dt$.

**Example**: find $i(t)$, $v(t)$



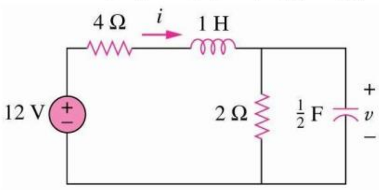

*Solution:*

Get $x(0)$.




$t < 0$

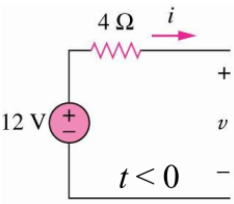
Get x(∞), $dx(0)/dt$, $s_{1,2}$, $A_{1,2}$.

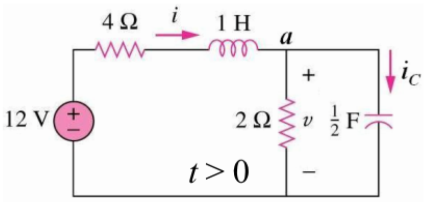



$t > 0$

**Steps:**

**Method (a)**




Initial conditions:

$$\begin{cases} v(0^+) = v(0^-) = 12\text{ V} & \text{(1a)} \\ i(0^+) = i(0^-) = 0 & \text{(1b)} \end{cases}$$

Applying KCL at node $a$ ($t > 0$),

$$i(0^+) = i_C(0^+) + \frac{v(0^+)}{2}$$

$$\Rightarrow i_C(0^+) = -6\text{ A}$$

$$\Rightarrow \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = -12\text{ V/s} \quad \text{(1c)}$$

Final values for $t \to \infty$:

$$\begin{cases} i(\infty) = \dfrac{12}{4+2} = 2\text{ A} \\ v(\infty) = 2i(\infty) = 4\text{ V} \end{cases}$$

Applying KCL at node $a$ gives

$$i = \frac{v}{2} + \frac{1}{2}\frac{dv}{dt} \qquad (2)$$

Applying KVL to the left mesh gives

$$4i + 1\frac{di}{dt} + v = 12 \qquad (3)$$

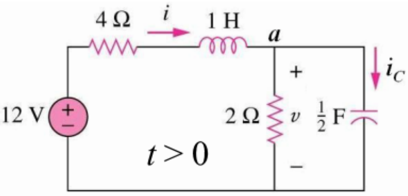
Substituting (2) into (3) gives

$$2v + 2\frac{dv}{dt} + \frac{1}{2}\frac{dv}{dt} + \frac{1}{2}\frac{d^2v}{dt^2} + v = 12$$

$$\Rightarrow \frac{d^2v}{dt^2} + 5\frac{dv}{dt} + 6v = 24 \quad (4)$$

Characteristic equation :

$$s^2 + 5s + 6 = 0$$




$$\Rightarrow s = -2, -3$$

$$v(t) = v_{ss} + v_t(t)$$

where $\begin{cases} v_{ss} = v(\infty) = 4 \\ v_t(t) = A_1 e^{-2t} + A_2 e^{-3t} \end{cases}$

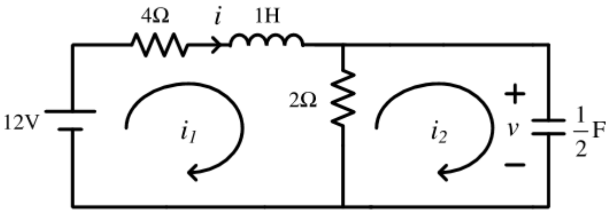
From (1a) and (1c) we obtain

$$\Rightarrow A_1 = 12, \; A_2 = 8$$

$i(t)$ can be obtain by using (2)

## Method (b) Using Mesh Analysis

$t > 0$




$$1\frac{di}{dt} + 4i + (i_1 - i_2)2 = 12 \qquad ......(A)$$

$$(i_2 - i_1)\times 2\Omega + \frac{1}{1/2}\int^t i_2 dt = 0 \quad ......(B)$$

From (B) , $2\frac{di_2}{dt} - 2\frac{di_1}{dt} + 2i_2 = 0$

$$\frac{di}{dt} + 6i_1 - 2i_2 = 12$$

$$-2\frac{di_1}{dt} + 2\frac{di_2}{dt} + 2i_2 = 0$$

12V

Eliminate $i_2$ variable from (C)

$$\frac{d^2i_1}{dt^2} + 5\frac{di_1}{dt} + 6i_1 = 12$$

$$\therefore i_1(t) = 2 - 6e^{-2t} + 4e^{-3t}, t \ge 0$$

Or eliminate $i_1$ variable from (C)

$$\frac{d^2i_2}{dt^2} + 5\frac{di_2}{dt} + 6i_2 = 0$$

$$\therefore i_2(t) = -12e^{-2t} + 6e^{-3t}, t \ge 0$$

Initial Condition , $i_1(0^+) = i(0^+) = i(0^-) = 0\text{A}$

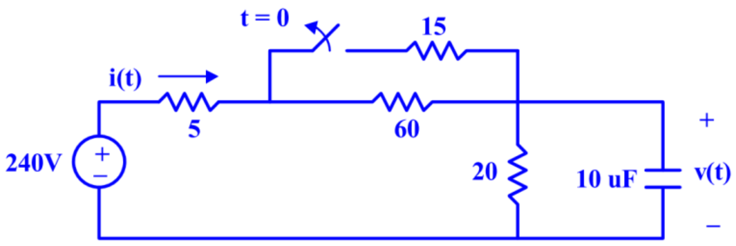
$$i_2(0^+) = i_c(0^+) = -6\text{A}$$

$$\frac{di_1(0^+)}{dt} = \frac{v_L(0^+)}{L} = 0$$

$$\frac{di_2(0^+)}{dt} = 0A/S$$

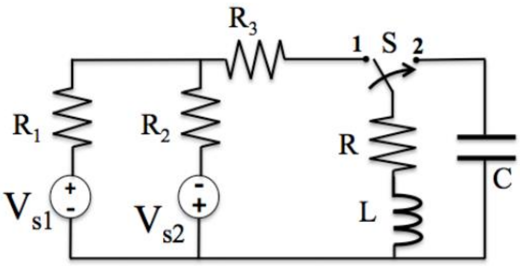
## 4. PROBLEMS

**Exercise 1.**




Find v(t) and i(t) for t > 0.

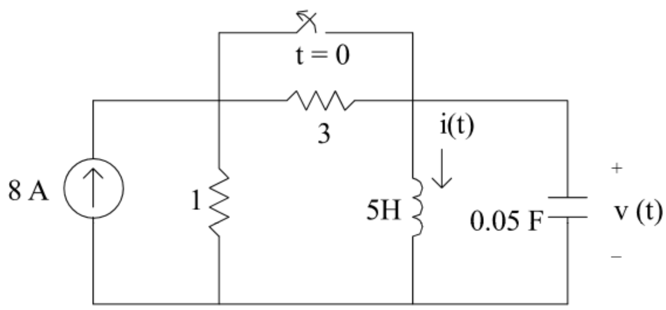
**Exercise 2.**




Consider the circuit shown in Fig. Give Vs1 = 50V, R1 = 5 Ω, Vs2 = 30V, R2 = 10 Ω, R3 = 4 Ω, R = 4 Ω, L = 2mH, C = 5mF. Determine the transient currents and voltages of the inductor L and capacitor C.

**Exercise 3.**

Determine i(t) in the circuit shown below for t > 0




**Exercise 4.**

Let us determine the output voltage v(t). R1 = 10 Ω, R2 =2 Ω, L = 2 H, C = 0.25 F