Rings and Fields

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Introduction

- We have studied sets with a single binary operation satisfying certain axioms
- What about two or more operations?
  $\rightarrow$ **Define Rings and Fields**

Applications

- QR Code, RSA cryptography, encodage/decodage systems, etc.
- Theorems: Cancellation, Divisors, etc.
- Algorithms: Euclidean algorithms, etc.
- Fundamental algebra in high school

Definition

A non-empty set with two binary operations $(R, +, .)$ such that

$$f : R \times R \to R, f(a, b) = a + b$$

$$g : R \times R \to R, g(a, b) = a.b$$

- $(R, +)$ is an abelian group under addition
- multiplication is associative $(ab)c = a(bc)$ for $a, b, c \in R$
- multiplication is distributive with respect to addition for $a, b, c \in R$

$$(a + b)c = ac + bc$$

$$a(b + c) = ab + ac$$

Definition

- If multiplication is also commutative, then the ring can be called a **commutative ring**.
- In a ring, multiplicative inverses are not required to exist.
- The **unit elements** in a ring have an inverse under multiplication.

Notation

- substraction: we write - b as shorthand for a + (-b).
- division: we write a/b as shorthand for a . (1/b) when 1/b exists.

Example 1

Are $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ rings under addition and multiplication?

Example 2

Is $\mathbb{N}$ a ring under addition and multiplication?

Example 3

Why $\mathbb{Z}_{12}$ is a ring?

Example 4

Any polynomial function is a ring.

Example 5

Is $(\mathbb{Z}, +, \min)$ a ring?

Example 6

The 2×2 matrices with entries in $\mathbb{R}$ form a ring under the usual operations of matrix addition and multiplication. But is it commutative?

Proposition 1

Let R be a ring with $a, b \in R$ then

- a0 = 0a = 0
- a(-b) = (-a)b = -(ab)
- (-a)(-b) = ab

Ring Homorphisms

if R and S are rings, then a ring homomorphism is a map $\phi : R \to S$ satisfying

- $\phi(a + b) = \phi(a) + \phi(b)$
- $\phi(ab) = \phi(a)\phi(b)$

for all $a, b \in$ R, if $\phi : R \to S$ is a one-to-one and onto homomorphism, then ϕ is called an isomorphism of rings.

Ring Homorphisms

Example 1

For any integer n we can define a ring homomorphism $\phi : \mathbb{Z} \to \mathbb{Z}_n$ by $a \to a \pmod{n}$

Definition 2

Let R be a ring and S is a subset of R, then S is a sub-ring of R if and only if

- $S \neq \emptyset$
- $ab \in S$ for all $a, b \in S$
- $a - b \in S$ for all $a, b \in S$

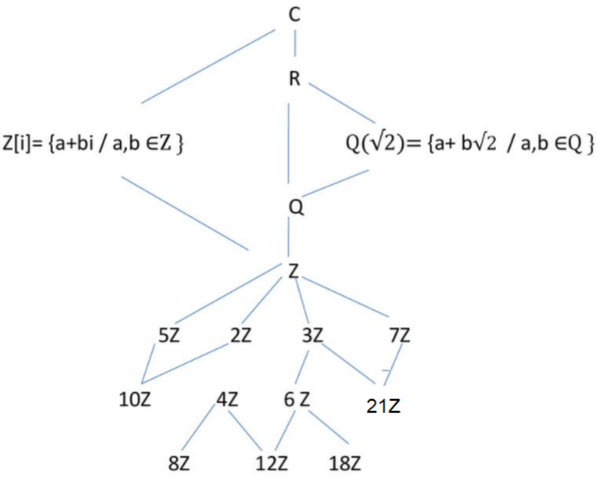
Example 1

$\mathbb{Z}$ and $\mathbb{Q}$ are subrings of $\mathbb{R}$;

Example 2

$n\mathbb{Z} = \{nk | k \in \mathbb{Z}\}$ is a subring of $\mathbb{Z}$ for any $n \in \mathbb{N}$;

SubRings

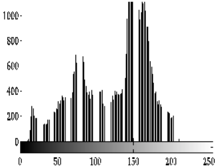


Ring Theory

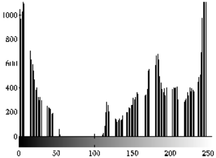
- Ring properties are used to define Integral domains and Fields.
- Ring theory in image segmentation: "Application of the Ring Theory in the Segmentation of Digital Images" the equivalence between two images A and B $\in G_{k\times m}(\mathbb{Z}_n)(+,.)$ is A $=$ S $+$ B (where S is a scalar image)

Rings

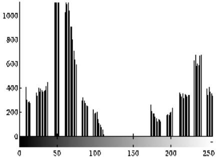
(a) Original Image



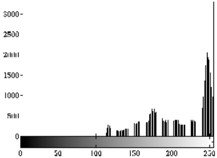
(b) Original histogram



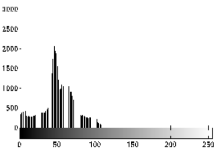
(c) Ring addition



(d) Ring subtraction



(e) Classical addition



(f) Classical subtraction

Zero Divisor

Definition 1

If R is a ring and r is a nonzero element in R, then r is said to be a **zero divisor** if there is some nonzero element $s \in R$ such that rs $=$ 0.

Alternative definition

If a, b are two ring elements with $a, b \neq 0$ but $ab = 0$ then a and b are called zero-divisors/divisor of zero.

Zero Divisor

Example 1

In $\mathbb{Z}_6$, we have $2.3 = 0$ so 2 and 3 are zero-divisors.

Example 2

In $\mathbb{Z}_{20}$, we have $4.5 = 2.10 = 0$ so 2, 4, 5, 10 are zero-divisors.

Zero Divisor

Proposition 1

For x be a ring element, x cannot be both invertible and a zero-divisor.
Proof: ?

Integral domains

Definition 2

An **integral domain** is a commutative ring with an identity $(1 \neq 0)$ with no zero-divisors.
That is $ab = 0 \rightarrow a = 0$ or $b = 0$.

Definition 3

If an element a in a ring R with identity has multipcalitive inverse, we say that a is a **unit**.

Definition 4

Characteristic of a ring R to be the least positive integer n such that nr $= 0$ for all $r \in R$. If no such integer exists, then the characteristic of R is defined to be 0.

Integral domains

Example 1

$\mathbb{Z}, \mathbb{R}, \mathbb{Q}$ are integral domains under addition and multiplication.

Example 2

$\mathbb{Z}_{13}$ is an integral domain.

Example 3

Is $(2\ \mathbb{Z}, +, .)$ is an integral domain?

Integral domains

Example 4

In the ring, $\mathbb{Z}_{20}$, the unit elements are $\{1, 3, 7, 11, 13, 17, 19\}$, the others are zero divisors

Example 5

$R = \mathbb{Z} \times \mathbb{Z}$ is a ring such that $x = (a, b)$, $y = (c, d) \in R$ then

- $x + y = (a + c, b + d)$
- $x . y = (a.c, b.d)$

Is R a ring? an integral domain?

Integral domains

Example 6

$\mathbb{Z}$ has the characteristic 0.

Example 7

$\mathbb{Z}_6$ has the characteristic 6 (because $6.5 = 0$).

Applications

- **Divisor definition**: Given elements a and b of R, one says that a divides b, or that a is a divisor of b, or that b is a multiple of a, if there exists an element x in R such that $ax = b$.
- **Euclidean algorithm** to find the greatest common divisor between two integers.
- The Fundamental Theorem of Algebra: A polynomial function of degree n has at most n solutions
- and more...

Integral domains

Theorems

Cancellation: Let D be an integral domain with $a, b, c \in D$. If $a \neq 0$ and ab = ac then b = c.
Prove: ??

Fields

Definition 1

A nonempty set R is a field if it has two closed binary operations: **addition** and **multiplication**

- both of which operations are **commutative, associative**,
- contain identity elements: 0 for addition, 1 for multiplication,
- contain inverse elements: -a for addition with $a \in R$, 1/a for multiplication with $a \in R$
- multiplication distributes over addition: for $a, b, c \in R$

$$(a + b)c = ac + bc$$

$$a(b + c) = ab + ac$$

Definition 2

If every nonzero element in a ring R is a unit, then R is called a **division ring**. A **commutative division ring** is called a **field**.

Definition 3

A subfield E of a field F is a subset of F that is a field with respect to the field operations of F.

Fields

Proposition 1

Let F be a field

- the additive identity is unique
- the additive inverse is unique
- the multiplicative identity is unique
- the multiplicative inverse is unique

Homomorphism

A **field homomorphism** is a function between two fields that preserves the field structure. A a map $\phi : F \to G$ between fields F and G is homomorphic if for all $a, b \in F$:

1. $\phi(a+b) = \phi(a) + \phi(b)$
2. $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$
3. $\phi(0) = 0 \quad \text{and} \quad \phi(1) = 1$

Example 1

Are $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ fields?

Example 2

The 2×2 matrices with entries in $\mathbb{R}$ form a field under the usual operations of matrix addition and multiplication?

Example 3

$\mathbb{Z}/p\mathbb{Z}$ is a field where p is prime number.

Example 4

Is any subset of $\mathbb{R}$ a field?

Fields

Example 5

The $\mathbb{F}_{256}$ is a finite field with 256 elements such that

- $256 = 2^8$, the base field $= \mathbb{F}_2$
- any element in $\mathbb{F}_{256}$ can be expressed by an 8-dimensional vector space over the $\mathbb{F}_{256}$
- any element can be written as follows: $a_7x^7 + a_6x^6 + \cdots + a_1x + a_0$ where $a_i \in \{0, 1\}$. Since there are $2^8 = 256$ different combinations of coefficients, there are exactly 256 elements.

Example 5

The $2\mathbb{F}_{256}$ is a finite field with 256 elements such that

- 0 (zero element)
- 1 (identity element)
- x
- $x + 1$
- x^2
- $x^2 + 1$
- $x^2 + x + 1$
- x^3
- $x^3 + x$
- $x^3 + x^2 + x + 1$
- ... (up to the 256th element)

Fields

Applications

- Define Vector Space over a field F
- Algorithm for QR code generations
- and more...