Introduction to Earth System

Dynamics of the atmosphere (cont.)

Note from 2025: to carefully check: https://rams.atmos.colostate.edu/at540/fall03/fall03Pt4.pdf

Holton J.R. An introduction to Dynamic Meteorology (4th Edition)

The equations

- The dynamics of the atmosphere → in the principles of conservation of momentum, mass, and energy
  - The Newton's equations of motion
  - The equation of continuity
  - The thermodynamic energy equation

$$\frac{du}{dt} - (f + \frac{u \tan \boldsymbol{\phi}}{a})v + \frac{1}{\boldsymbol{\rho}}\frac{\partial p}{\partial x} + F_x = 0$$

$$\frac{dv}{dt} + (f + \frac{u \tan \boldsymbol{\phi}}{a})u + \frac{1}{\boldsymbol{\rho}}\frac{\partial p}{\partial y} + F_y = 0$$

$$p = R\boldsymbol{\rho}T$$

$$\frac{\partial p}{\partial z} + g\boldsymbol{\rho} = 0$$

$$\frac{dT}{dt} + (\gamma - 1)T\nabla \cdot \mathbf{V} = \frac{Q}{c_p}$$

$$\frac{\partial \boldsymbol{\rho}}{\partial t} + \nabla \cdot \boldsymbol{\rho}\mathbf{V} = 0$$

$$\frac{\partial \boldsymbol{\rho}_w}{\partial t} + \nabla \cdot \boldsymbol{\rho}_w\mathbf{V} = [sources - sinks]$$

- **Independent variables:** space & time coordinates (x,y,z,t)
- **Dependent variables:** velocity, pressure, density, temperature

The equation of state

- The ideal gas law:

$$pV = nR^{*}T \quad (1)$$

R*: the universal gas constant
(=8.314 J/(mol.K))
n: number of moles of gas
T: absolute temperature

- The mean molecular weight of air is 29 → the air parcel's mass $m = \rho V = 29 \times n$
- Dividing Eq(1) by the volume V → the equation of state:

$$\boxed{p = R\rho T}$$

R=R*/29=287 J/(mol.K) is the gas constant for dry air

Q: Why the mean molecular weight of air is 29?

The motion equations

- Large scale (e.g. synoptic) motion systems in the troposphere:
  - Vertical scale: H= 10 km
  - Horizontal scale: L= 1000 km
  - A typical grid-box of an NWP ~ 10 km x 10 km x 100m

The motion equations

1. The pressure force per unit mass is:

$$\boxed{\boldsymbol{F}_p = \left(-\frac{1}{\rho}\frac{\partial p}{\partial x}, -\frac{1}{\rho}\frac{\partial p}{\partial y}, -\frac{1}{\rho}\frac{\partial p}{\partial z}\right) = -\frac{1}{\rho}\nabla p}$$

2. The force due to gravity → vertically downward (to the earth's center): $\mathbf{g}^* = -g\mathbf{k}$ (the star on g* will be described later)

3. The force of friction → opposite direction to the flow velocity

$\mathbf{F}_f = -\kappa\mathbf{V}$ where κ is the friction coefficient, that depend on location & could be also on velocity

The motion equations in an Inertial frame of reference

- Inertial frame of reference: a frame of reference that is not undergoing any acceleration

- The basic equations of motion according to the 2nd Newton's law (**a**=**F**/m):

$$\boxed{\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^* + \mathbf{F}_f}$$

- Recall the continuity equation in the Lagrangian form

$$\boxed{\frac{d\rho}{dt} + \rho\nabla.\mathbf{V} = 0}$$

The motion equations in an Inertial frame of reference

$$\boxed{\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^* + \mathbf{F}_f}$$

$$\boxed{\frac{d\rho}{dt} + \rho\nabla.\mathbf{V} = 0}$$

If the fluid is
- incompressible
- inviscid (i.e. no friction)

→

$$\boxed{\frac{d\mathbf{V}}{dt} = -\frac{1}{\rho}\nabla p + \mathbf{g}^*}$$

$$\boxed{\nabla.\mathbf{V} = 0}$$

Written in cartesian coordinates, we get:

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = -\frac{1}{\rho}\frac{\partial \rho}{\partial x}$$

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v = -\frac{1}{\rho}\frac{\partial \rho}{\partial y}$$

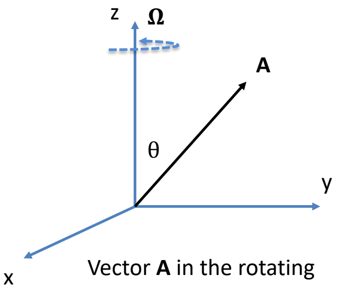
$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w = -\frac{1}{\rho}\frac{\partial \rho}{\partial z} - g$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

The motion equations in a rotating coordinate frame

Theorem: **A** is a vector fixed in a rotating frame with the constant angular velocity *Ω*. We have:

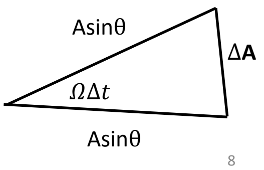
$$\boxed{\frac{d\mathbf{A}}{dt} = \mathbf{\Omega}\times\mathbf{A}}$$



Vector **A** in the rotating coordinate frame

Exercise #1: prove the above theorem

Solution
- $\mathbf{A}_\Omega$ The projection of **A** on the **Ω**-axis does not change
- $\mathbf{A}_{XY}$ The projection of **A** on the X-Y plane is Asinθ, which does not change in magnitude, but changes in direction
- Δ**A** is on the the X-Y plane → perpendicular to **Ω**;
  Δ**A** is perpendicular to **A**
  → Δ**A**=ΔA**n**
  where **n** is a unit vector perpendicular to both **Ω** and **A**



$$\rightarrow \frac{d\mathbf{A}}{dt} = \frac{d\mathbf{A}_\Omega}{dt} + \frac{d\mathbf{A}_{XY}}{dt} = \frac{d\mathbf{A}_{XY}}{dt} \approx \lim_{\Delta t\to 0}\frac{\Delta\mathbf{A}}{\Delta t} = A\sin\theta\Omega\mathbf{n} = \mathbf{\Omega}\times\mathbf{A}$$

The motion equations in a rotating coordinate frame

- If **A** is not fixed in the rotating frame
- There is the following relationship between the rate of change of **A** in the absolute frame and the rotating frame:

$$\boxed{\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{A}}$$

Exercise #2: Prove the above relationship

Hint: Consider a cartesian coordinates in the rotating frame

The motion equations in a rotating coordinate frame

$$\boxed{\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{A}}$$

Applications:

- If **A** is the position vector **r** → $\left(\frac{d\mathbf{r}}{dt}\right)_{\mathrm{I}} = \mathbf{V}_{\mathrm{I}}$ & $\left(\frac{d\mathbf{r}}{dt}\right)_{\mathrm{R}} = \mathbf{V}_{\mathrm{R}}$

→ $$\boxed{\mathbf{V}_{\mathrm{I}} = \mathbf{V}_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{r}}$$

$$\underbrace{\mathbf{V}_{\mathrm{I}}}_{\text{Inertial (absolute) velocity}} = \underbrace{\mathbf{V}_{\mathrm{R}}}_{\text{Relative velocity}} + \underbrace{\mathbf{\Omega}\times\mathbf{r}}_{\text{Velocity of the frame}}$$

Exercise #3:

1. What is the absolute velocity of USTH (consider as a point, lat=21.05ºN, lon=105.81ºE)?
2. Prove that the value of the velocity due to the earth's rotation at 60ºN is half of that at the equator

The motion equations in a rotating coordinate frame

Relative acceleration

- From $\boxed{\left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{A}}{dt}\right)_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{A}}$ & $\boxed{\mathbf{V}_{\mathrm{I}} = \mathbf{V}_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{r}}$
- Let **A** be $\mathbf{V}_{\mathrm{I}}$

→ $$\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + \left(\frac{d\mathbf{\Omega}\times\mathbf{r}}{dt}\right)_{\mathrm{R}} + \mathbf{\Omega}\times\mathbf{V}_{\mathrm{R}} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})$$

→ $$\boxed{\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + 2\mathbf{\Omega}\times\mathbf{V}_{\mathrm{R}} + \underbrace{\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r})}_{\text{Centrifugal acceleration}}}$$

The centrifugal acceleration depends only on position → combine with the gravitational acceleration, we get an apparent gravitational acceleration

$$\boxed{\mathbf{g} = \mathbf{g}^{*} - \mathbf{\Omega}\times\mathbf{\Omega}\times\mathbf{r}}$$

Exercise #4

- Estimate the centrifugal acceleration at the Equator & compare it to the gravitational acceleration value.

Results: the centrifugal acceleration ~0.3% the gravitational acceleration

→ The flattened form of the earth
→ You lose weight when you travel to the lower latitude ☺

The Coriolis acceleration

$$\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} \underbrace{+ 2\boldsymbol{\Omega}\times\mathbf{V}_{\mathrm{R}}}_{\text{Coriolis acceleration}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

- **The Coriolis acceleration:**
  - No component in the direction of motion
  - Varies linearly with the motion speed
  - Perpendicular to the velocity
  - An important factor in all large-scale weather systems
  - When the air is moving → deflect the direction → explain the rotational character of the atmospheric flow

Exercise #5:

Estimate the deflection of a tropical cyclone at 30ºN, travelling for 1 hours at the speed of 100 km/h

Motion equations in component form

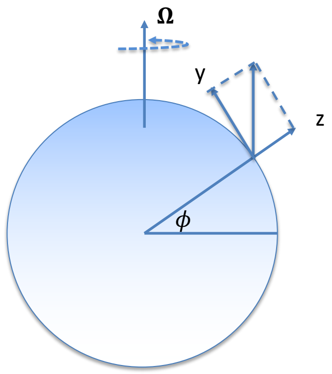
$$\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} = -\frac{1}{\rho}\nabla p + \mathbf{g}^{*} + \mathbf{F}_{f}$$

In the inertial frame

$$\left(\frac{d\mathbf{V}_{\mathrm{I}}}{dt}\right)_{\mathrm{I}} = \left(\frac{d\mathbf{V}_{\mathrm{R}}}{dt}\right)_{\mathrm{R}} + 2\boldsymbol{\Omega}\times\mathbf{V}_{\mathrm{R}} + \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{r})$$

- We assume there is no friction for instance → the motion equation in the rotating frame (**V**= $\mathbf{V}_{\mathrm{R}}$) is:

$$\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\times\mathbf{V} + \frac{1}{\rho}\nabla\mathrm{p} - \mathbf{g} = 0$$



- We will write the above equation in the local cartesian coordinates (x,y,z)

$$\mathbf{V} = (u, v, w)$$

$$\nabla\mathrm{p} = \left(\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}\right)$$

$$\boldsymbol{\Omega} = (0, \Omega\cos\emptyset, \Omega\sin\emptyset)$$

? $2\boldsymbol{\Omega}\times\mathbf{V}$ $= (2\mathrm{w}\Omega\cos\emptyset - 2\mathrm{v}\Omega\sin\emptyset, 2\mathrm{u}\Omega\sin\emptyset, -2\mathrm{u}\Omega\cos\emptyset)$

  - Assuming w is much smaller than u, v → neglect the term $2\mathrm{w}\Omega\cos\emptyset$
  - Let $f = 2\Omega\sin\emptyset$ → is called the Coriolis parameter

Motion equations in component form

$$\boxed{\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\times\mathbf{V} + \frac{1}{\rho}\nabla\mathrm{p} - \mathbf{g} = 0}$$

$$\boxed{2\boldsymbol{\Omega}\times\mathbf{V} \approx (-\mathrm{fv}, \mathrm{fu}, -2\mathrm{u}\Omega\cos\emptyset)}$$

- The horizontal components of the equation of motion become:

$$\boxed{\frac{du}{dt} - fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0}$$

$$\boxed{\frac{dv}{dt} + fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0}$$

- The vertical component of the equation of motion becomes:

$$\boxed{\frac{dw}{dt} - 2\mathrm{u}\Omega\cos\emptyset + \frac{1}{\rho}\frac{\partial p}{\partial z} + \mathrm{g} = 0}$$

- In case there is no motion
  → the hydrostatic equation

- The steady-state force balance between the Coriolis force & the pressure gradient force → **geostrophic balance**
- **Geostrophic balance** is the approximate state of most large-scale flows in the ocean and atmosphere

$$\boxed{-fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0}$$

$$\boxed{fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0}$$

Practice #6 with Python: Computing wind speed in a geostrophic balance

1. Given the earth divided into a horizontal mesh of 360 x 180 points
2. Randomly generate pressure at 5km height in the atmosphere, within the values ranging from 500hPa to 600hPa
3. Use differences between adjacent cells to estimate pressure gradients.
4. Compute the Coriolis parameter at each point, plot the values ($f = 2\Omega\sin\emptyset$)
5. Assuming having the geostrophic equations, equal density, please compute horizontal wind components.
6. Plot wind vectors

Bonus: download the NCEP reanalysis monthly pressure at the tropopause level https://downloads.psl.noaa.gov/Datasets/ncep.reanalysis/Monthlies/tropopause/pres.mon.mean.nc and compute horizontal wind speed

$$\boxed{-fv + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0}$$

$$\boxed{fu + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0}$$