Mathematics for Physicists
Lecture 1
Line integrals

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Contents

Basic facts on paths

Definition 1.1

A path of $\mathbb{R}^n$ *is a map* $c : I \to \mathbb{R}^n$, *with* $I = [a, b]$. *The subset of* $\mathbb{R}^n$ $\mathcal{C} = c([a, b])$ *is called the curve parametrized by the path* c. $c(a)$ *and* $c(b)$ *are the enpoints of the curve* $\mathcal{C}$. *We way also that* c *is a parametrization of the curve* $\mathcal{C}$.

Example 1.1

Let $c : [0, 1] \to \mathbb{R}$ *be given by*

$$c(t) = (x_0, y_0, z_0) + tv$$

where (x_0, y_0, z_0) *is a fixed point of* $\mathbb{R}^3$ *and* v *a non null vector of* $\mathbb{R}^3$. *Then the curve associated with this path* c *is the segment of* $\mathbb{R}^3$ $[(x_0, y_0, z_0); (x_0, y_0, z_0) + v]$.

Example 1.2

Let $c : [0, 2\pi] \to \mathbb{R}^2$ *be given by*

$$c(t) = (\cos t, \sin t)$$

Then the associated curve $\mathcal{C}$ *is the unit circle of* $\mathbb{R}^2$.

Definition 1.2

If c *is a continuous map, derivable (or differentiable)* ..., *we say that the path* c *is continuous, derivable,* .... *The velocity vector at time* t *at point* $c(t)$ *is the vector* $c'(t)$. *The speed at time* t *and point* $c(t)$ *is the norm of this vector, that is* $\| c'(t) \|$.

Remark 1.1

The velocity $c'(t)$ *is a vector tangent to the path* c. *We say also that it is tangent to the curve* $\mathcal{C}$ *parametrized by* c.

Path integrals

Definition 1.3

Let $c : [a, b] \to \mathbb{R}^n$ be a path of class C^1 (by pieces). We call length of c the number

$$L(c) \equiv \int_a^b \| c'(t) \| \, dt$$

More generally, we have the following definition

Definition 1.4

With the same previous assumptions, let moreover $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function (at least in a neighborhood of $\mathcal{C}$). Then we define

$$\int_c f(s)ds \equiv \int_a^b f(c(t)) \| c'(t) \| \, dt$$

If $f = 1$, we recover the length of the path c.
Explanations : work with a path $c : [a, b] \to \mathbb{R}^3$. Make polygonal approximations.
Subdivision of order N of $[a, b]$. Consider the polygonal lines based on points $c(t_i)$, where

$$a = t_0 < t_1 < \ldots < t_N = b, \; t_{i+1} - t_i = \frac{b-a}{N}, 0 \leq i \leq N-1$$

Length of c will be almost equal to the length of this broken line, for N large enough, that is

$$S_N = \sum_{i=0}^{N-1} \| c(T_{i+1}) - c(t_i) \|$$

If $c(t) = (x(t), y(t), z(t))$, for $x(t)$, $y(t)$ and $z(t)$ on the interval $[t_i; t_{i+1}]$, there exists t_i^*, t_i^{**}, t_i^{***} such that

$$x(t_{i+1}) - x(t_i) = x'(t_i^*)(t_{i+1} - t_i)$$

$$y(t_{i+1}) - y(t_i) = y'(t_i^{**})(t_{i+1} - t_i)$$
$$z(t_{i+1}) - z(t_i) = z'(t_i^{***})(t_{i+1} - t_i)$$

And thus, we obtain

$$S_N = \sum_{i=0}^{N-1} \sqrt{x'(t_i^*)^2 + y'(t_i^{**})^2 + z'(t_i^{***})^2}(t_{i+1} - t_i)$$

Here is an important case. Assume that $\mathcal{C}$ is a plane curve, for example contained in the plane $(x0y)$ of $\mathbb{R}^3$. Let f be a positive function of variables x and y. Then $\int_c f(s)ds$ is the area of the lateral surface.

Line integrals

$\vec{F}$ a force field of $\mathbb{R}^3$ that is a map from $\mathbb{R}^3$ into $\mathbb{R}^3$.
Assume that this field acts on a given particle of $\mathbb{R}^3$ and this particle is moving along a fixed curve $\mathcal{C}$ and that is submitted to this force field $\vec{F}$.
$\rightarrow$ compute the work done by this field $\vec{F}$ on this particle.
If $\mathcal{C}$ is a line segment, for example, directed by a vector $\vec{d}$, with $\| \vec{d} \|$ being the length of $\mathcal{C}$, then the corresponding work is given by

$$T_{\mathcal{C}}(\vec{F}) = \vec{F}.\vec{d}$$

If now, $\mathcal{C}$ is no more a line segment,

$$T_{\mathcal{C}}(F) \equiv \int_a^b \vec{F}(c(t)).c'(t)dt$$

where c is a parametrization of the curve $\mathcal{C}$ and giving the position of the particle on the curve $\mathcal{C}$.

Explanation : note that in the case where $\mathcal{C}$ is a line segment, we recover the first formula. Otherwise, $t \in [t, t + \Delta t]$ with Δt small. Then the particle will move from $c(t)$ to $c(t + \Delta t)$. The corresponding displacement vector is thus

$$\vec{\Delta s} = c(t + \Delta t) - c(t) \simeq c'(t)\Delta t$$

The work between $c(t)$ and $c(t + \Delta t)$ will then be almost equal to

$$\vec{F}(c(t)).\vec{\Delta s} \simeq \vec{F}(c(t)).c'(t)\Delta t$$

Now if we divide the interval $[a, b]$ along a regular subdivision of order N as above, with $\Delta t = t_{i+1} - t_i$, the total work done by $\vec{F}$ will be almost equal to

$$\sum_{i=0}^{N-1} \vec{F}(c(t_i)).\vec{\Delta s} \simeq \sum_{i=0}^{N-1} \vec{F}(c(t_i)).c'(t_i)\Delta t$$

Definition 1.5

Let $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a force field, continuous in a neighborhood of the path c, $c : [a, b] \to \mathbb{R}^n$ (class C^1). Then we define the line integral of $\vec{F}$ along c, or in other words, the work of $\vec{F}$ on c as the number defined by

$$\int_c \vec{F}.ds \equiv \int_a^b \vec{F}(c(t)).c'(t)dt$$

Example 1.3

$c(t) = (\sin t; \cos t, 0)$, $0 \leq t \leq 2\pi$, *et* $F = (x, y, z)$.

Definition 1.6

Notation : With the above assumptions and notations, we also denote, if $\vec{F} = (F_1, F_2, F_3)$, and $n = 3$

$$\int_c \vec{F}.ds = \int_c F_1 dx + F_2 dy + F_3 dz$$

Thus if $c(t) = (x(t), y(t), z(t))$, we have

$$\int_c F_1 dx + F_2 dy + F_3 dz =$$

$$\int_a^b [F_1(c(t))x'(t) + F_2(c(t))y'(t) + F_3(c(t))z'(t)]dt$$

Example 1.4

Compute $\int_c x^2 dx + xydy + dz$ *with* $c(t) = (t, t^2, 1)$ *and* $0 \leq t \leq 1$.

Example 1.5

Compute $\int_c \cos z dx + e^x dy + e^y dz$ *with* $c(t) = (1, t, e^t)$ *and* $0 \leq t \leq 2$.

$\int_c \vec{F}.ds$ depends on $\vec{F}$ but also on the path c.
What is the link with the associated curve $\mathcal{C}$?
If c_1 and c_2 are two different paths, but parametrizing the same curve $\mathcal{C}$, we do not have

$$\int_{c_1} \vec{F}.ds = \int_{c_2} \vec{F}.ds \ ?$$

However, there are cases where

$$\int_{c_1} \vec{F}.ds = \mp \int_{c_2} \vec{F}.ds$$

Definition 1.7

Let $h : I \to I_1$ be a C^1 bijective map with $I = [a, b]$ and $I_1 = [a_1, b_1]$. Let $c : I_1 \to \mathbb{R}^n$ be a path of class C^1 (by pieces). Then $p = c \circ h : I \to \mathbb{R}^n$ is a parametrization of c.

In fact we have $Im\ c = \ Im\ p$. Thus c and p are two parametrizations of the same curve $\mathcal{C}$.

As we have $p'(t) = c'(h(t)).h'(t)$, we note that :
1) if h is stricly increasing, $h(a) = a_1$ and $h(b) = b_1$.
2) if h is strictly decreasing, $h(a) = b_1$ and $h(b) = a_1$.
Thus, we have one of the two following cases :
1) $c \circ h(a) = c(a_1)$ *and* $c \circ h(b) = c(b_1)$
2) or $c \circ h(a) = c(b_1)$ *and* $c \circ h(b) = c(a_1)$
In the first case, we say that the re-parametrization preserves the orientation, while in the second case, it reverses the orientation.
If we preserve the orientation, a particle moving on $\mathcal{C}$ according to $c \circ h$ will move in the same direction as along c.

Example 1.6

If $c : [a, b] \to I\!R^3$ *is* C^1 *then*

$$c_{op} : [a, b] \to I\!R^3, c_{op}(t) = c(a + b - t)$$

is a reparametrization of c *reversing the orientation.*

Theorem 1.1

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a vector field, continuous in a neighborhood of the path of class C^1 $c : [a, b] \to \mathbb{R}^n$. Let $p : [a_1, b_1] \to \mathbb{R}^n$ be a reparametrization of c. Then

$$\int_p F.ds = \mp \int_c F.ds$$

with the sign $+$ if p preserves the orientation of c and the sign $-$ if p reverses the orientation.

Example 1.7

Let $f(x, y, z) = (yz, xz, xy)$ and

$$c : [-5; 10] \to \mathbb{R}^3, \; c(t) = (t, t^2, t^3)$$

We find

$$\int_c F.ds = 984,375 \text{ and } \int_{c_{op}} F.ds = -984,375$$

Theorem 1.2

(change of parametrization for the path integrals). Let c be a path of class C^1 (by pieces), p any reparametrization of c and $f : \mathbb{R}^3 \to \mathbb{R}$ a continuous map. Then

$$\int_c f ds = \int_p f ds$$

In the case when the field is a gradient, we have

Theorem 1.3

Assume that $\vec{F} = \nabla f$, with $f : \mathbb{R}^3 \to \mathbb{R}$ C^1. Then

$$\int_c F.ds = f(c(b)) - f(c(a))$$

Example 1.8

Let $c(t) = (4^4/4, \sin^3(t\pi/2), 0)$, $t \in [0, 1]$. *We want to compute* $\int_c ydx + xdy$. *Here* $F = (y, x, 0)$ *and thus* $F = \nabla f$ *avec* $f = xy$. *Thus ....*

Definition 1.8

We define a simple curve $\mathcal{C}$ *as being the image of a path of class* C^1 *(by pieces)* $c : I \to \mathbb{R}^n$ *injective on the interior of* I. *We then say that* c *is an adapted parametrization to* $\mathcal{C}$

A simple curve corresponds to a curve which does not self intersect except eventually at the endpoints.

If $I = [a, b]$, $c(a) = P$ and $c(b) = Q$ are the endpoints of $\mathcal{C}$. Any simple curve has two possible orientations. A curve equipped with one of these two orientations is called an oriented simple curve.

Definition 1.9

We say that a simple curve $\mathcal{C}$ is closed if moreover we have $c(a) = c(b)$. Any simple and closed curve has two possible orientations.

Definition 1.10

Let $\mathcal{C}$ be a simple orientated curve (eventually closed). Then we set

$$\int_{\mathcal{C}} F.ds = \int_{c} F.ds$$

where c is any parametrization but adapted and preserving the orientation of $\mathcal{C}$.

Be careful : we need to check each time that the parametrization is adapted and preserves the orientation.

Example 1.9

Let $c(t) = (\cos t, \sin t, 0)$ *and* $p(t) = (\cos 2t, \sin 2t, 0)$,
$0 \leq t \leq 2\pi$, *and* $F = (y, 0, 0)$. *Then we find that*

$$\int_c F.ds = -\pi \text{ but } \int_p F.ds = -2\pi$$

Note that Im $c =$ *Im* p. p *is not injective.*

Proposition 1.1

Let $\mathcal{C}^-$ *the same curve as* $\mathcal{C}$ *but with the opposite orientation. Then*

$$\int_{\mathcal{C}} F.ds = -\int_{\mathcal{C}^-} F.ds$$

Remark 1.2

We may generalize all these facts to "sums" of oriented maps.