Probability and Statistics
Lecture 2
Random variables

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Plan

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2. Mass and density functions
3. Important examples of discrete rv (drv)
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   - Bernoulli trials and Binomial Law
   - Geometric Law
   - Poisson law
4. Examples of continuous rv
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   - Gamma law
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5. Expectation and variance

A random variable : quantity which depends only upon the result of the (random) experiment :
- number of "6" obtained when throwing 3 dices ;
- number of phone calls during one hour ...

Definition 1.1

Let $(\Omega, \mathcal{T}, P)$ be a probability space. Let $E \subset \mathbb{R}^d$ be given. Any map

$$X : \Omega \to E, \omega \in \Omega \to X(\omega)$$

is called a random variable on the probability space, valued in E.

If $E \subset \mathbb{R}$, we say that is a real random variable ;
If $E \subset \mathbb{R}^d$, we say that it is a random vector.
So be careful : a random vector is a map.

Distribution function of rv

Definition 2.1

Distribution function of the real rv X :

$$F_X(x) = P(X \leq x), \ \forall x \in \mathbb{R}$$

Proposition 2.1

Properties : *we have*

1. $0 \leq F_X(x) \leq 1, \ \lim_{+}\infty F_X = 1, \ \lim_{-\infty} F_X = 0\,;$
2. *F_X is increasing (non decreasing)*
3. *F_X is right continuous, that is $F_X(x) = F_X(x^+)$ (right limit at x).*

Proposition 2.2

We have

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

Proposition 2.3

We have

$$P(X = x) = F_X(x) - F_X(x^-)$$

wher $F_X(x^-)$ is teh left limit at x.

Important : if F_X is a continuous function, then the probability $P(X = x)$ is zero for any real x.
In that case, that is if the distribution function is continuous, we have

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b)$$

Definition 2.2

Let X be a rv taking at most a countable number of values, $X(\Omega) = \{x_1, x_2, ...\}$. Then we say that X is a discrete rv.

Definition 2.3

Let X taking an infinite non countable number of values. If F_X is continuous, we say that X is a continuous rv.

Mass and density functions

Definition 3.1

Let X be a discrete rv, $X(\Omega) = \{x_1, x_2, ...\}$. p_X :

$$p_X(x_k) = P(X = x_k), \ k = 1, 2, ...$$

= mass function of X = probability law of X.

Definition 3.2

Let X be a continuous rv. The function f_X defined by (if it exists)

$$f_X(x) = \frac{d}{dx} F_X(x)$$

is called the probability density function of X.

Thus $f_X(x)dx \sim$ probability that rv X takes value x in a small interval of length dx around x.

Properties :

i) $f_X(x) \geq 0$, because F_X is increasing ;

ii)

$$\int_{-\infty}^{x} f_X(t)dt = F_X(x)$$

$->$

$$\int_{-\infty}^{+\infty} f_X(x)dx = F_X(+\infty) = 1$$

A positive function with this property is called a density (function).

Important property

$$P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$$

The probability that X belongs to the interval $(a, b]$ is given by the area under the curve of f_X from a to b.

Drv/Bernoulli law

Random experiment E. Let $A \subset \Omega$ and let X be the indicator rv of this event A, that is $X = 1$ if we get A and 0 otherwise.
We say that X follows a Bernoulli law with parameter p, where $p = P(A)$ is the probability of a success. We have

x	0	1	Σ
$p_X(x)$	$1-p$	p	1

We may also write

$$p_X(x) = p^x q^{1-x}, x = 0, 1$$

where $q = 1 - p$. Here p is a known parameter.

DRV/Binomial laws

We repeat the previous experiment E n times.
We then say that the trials E_1, ..., E_n form Bernoulli trials if
a) these trials are independent and
b) the probabillity of a success is the same for each trial.

Binomial law

X : number of success with n Bernoulli trials.
X follows a bionomial law with parameters n and p, where p is the probability of one success. We have $X(\Omega) = \{0, 1, ..., n\}$. We write $X \sim B(n, p)$.
Probability law of $X \sim B(n, p)$:

$$p_X(k) = C_k^n p^k q^{n-k}, \text{ for } k = 0, 1, ..., n$$

Geometric Law

X : number of necessary Bernoulli trials in order to obtain the first success. Then $X(\Omega) = \{1, 2, ...\}$.
X follows a geometric law with parameter p :$X \sim Geom(p)$.
In that case, we have

$$p_X(k) = q^{k-1}p, \quad \text{pour } k = 1, 2, ...$$

We have :

$$F_X(n) = ... = 1 - q^n \text{ et } P(X > n) = q^n, n = 1, 2, ...$$

As a last remark, it may happen that we can also define another X denoted by Y, to be the number of Bernoulli trials before getting the first success. In that case $Y(\Omega) = \{0, 1, ..\}$ and function p_Y becomes

$$p_Y(k) = q^k p, \ k = 0, 1, ..$$

Then $Y = X - 1$.

Definition 4.1

Let X be a drv with $X(\Omega) = \{0, 1, ..; \}$ and

$$p_X(k) = \frac{e^{-\alpha}\alpha^k}{k!} \text{ for } k = 0, 1, ...$$

We then say that X follows a Poisson law with parameter $\alpha > 0$.
We write $X \sim Poi(\alpha)$.

CRV/ Uniform Law

We choose at random a number X in the interval $[a, b]$.
If the density function of X is given by

$$f_x(x) = \frac{1}{b-a}, a \leq x \leq b$$

we say that X follows a uniform law on the interval $[a, b]$. We write $X \sim U(a, b)$.
Distribution function of X :

$$F_X(x) = 0 \text{ if } x < a, \frac{x-a}{x-b} \text{ if } a \leq x \leq b, 1 \text{ if } x > b$$

If $[c, d] \subset [a, b]$, then

$$P(c < X \leq d) = \frac{d-c}{b-a}$$

The probability for X to be in a given sub interval only depends on the length of this interval.

CRV/ Exponential Law

We say that X defined on $[0, +\infty)$ follows an exponential law with parameter λ if

$$f_X(x) = \lambda e^{-\lambda x}, x \geq 0$$

We write $X \sim Exp(\lambda)$.
This is a density. The distribution function is given by

$$F_X(x) = e^{-\lambda x} \text{ if } x \geq 0 \text{ and } 0 \text{ if } x < 0$$

In particuliar,

$$P(X > 0) = e^{-\lambda x}, x \geq 0$$

.

CRV/ Gamma law

Gamma function $\Gamma(.)$:

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} e^{-x} dx$$

pour $\alpha > 0$.
I.B.P $->$

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1), \text{ if } \alpha > 1$$

If $\alpha = n = 2, 3, ...,$ then

$$\Gamma(n) = (n-1)!$$

Also :

$$\Gamma(1/2) = \sqrt{\pi}$$

Let X : a positive rv.
If its density function is given by :

$$f_X(x) = \frac{(\lambda x)^{\alpha-1}\lambda e^{-\lambda x}}{\Gamma(\alpha)} \text{ for } x \geq 0,$$

we say that X follows a
Gamma law wth parameters $\alpha > 0$ and $\lambda > 0$.
We write $X \sim G(\alpha, \lambda)$.

Remarks 5.1

i) α : shape parameter ; λ : scale parameter. As f_X has a shape which varies rapidly when parameter α takes different values, Gamma law is very often used in applications.
ii) Gamma law = exponential law when $\alpha = 1$.
iii) If $\alpha = n/2$, $n \in \mathbb{N}$ and $\lambda = 1/2$, Gamma law is also called khi-square law with 2 to n degrees of freedom.

CRV/ Gaussian law

Let X be a rv with $X(\Omega) = \mathbb{R}$. If the density function of X is of the form

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} exp\{-\frac{(x-\mu)^2}{2\sigma^2}\}, x \in \mathbb{R}$$

we say that X follows a
gaussian or normal law with parameters μ and σ^2 with $\sigma > 0$.
We write $X \sim N(\mu, \sigma^2)$.
It is also called Laplace-Gauss law.
Parameter μ is a position paramter, while σ is a scale parameter.
All gaussian laws have a bell shape.

If $\mu = 0$ and $\sigma = 1$: X follows a standard normal law. Its density function is given by

$$\phi(z) \equiv \frac{1}{\sqrt{2\pi}} exp(-z^2/2), z \in \mathbb{R}$$

and its distribution function denoted by Φ, is

$$\Phi(z) = \int_{-\infty}^{z} \phi(y) dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy$$

IF $X \sim N(\mu, \sigma^2)$, $->$ distribution function in terms of Φ :

$$F_X(x) = \Phi(\frac{x-\mu}{\sigma})$$

corresponding density function given by

$$f_X(x) = \frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})$$

That is we can obtain everything from the standard law $N(0, 1)$.

Expectation and variance

Definition 6.1

The expectation or the mean of a rv X, denoted by $E(X)$ or equivalently by $< X >$ is defined by

$$E(X) \equiv \mu_X = \begin{cases} \sum_{k=1}^{\infty} x_k p_X(x_k) \text{ if } X \text{ is discrete} \\ \\ \int_{-\infty}^{\infty} x f_X(x) dx \text{ if } X \text{ is continuous.} \end{cases}$$

The expectation is linear.

Proposition 6.1

Let X be a rv and $Y = g(X)$. Then we have

$$E(Y) = \begin{cases} \sum_{k=0}^{\infty} g(x_k) p_X(x_k) \text{ if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} g(x) f_X(x) dx \text{ if } X \text{ is continuous} \end{cases}$$

Definition 6.2

The variance of rv X is defined by

$$Var(X) = \sigma_X^2 = E((X - E(X)^2)$$

If X has a density f_X, then we have

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f_X(x)dx$$

with $\mu = E(X)$. The standard deviation is defined by

$$STD(X) = \sigma_X = \sqrt{Var(X)}$$

We have

$$Var(aX + b) = a^2 Var(X)$$

and

$$Var(X) = E(X^2) - (E(X))^2$$

Law	Parameters	Mean	Variance
Bernoulli	p	p	pq
Binomial	n et p	np	npq
Geometric	p	$1/p$	q/p^2
Poisson	α	α	α

Law	Parameters	Mean	Variance
Uniform	$[a, b]$	$(a+b)/2$	$(b-a)^2/12$
Exponential	λ	$1/\lambda$	$1/\lambda^2$
Gaussian	μ and σ^2	μ	σ^2

Theorem 6.1

Bienaymé-Tchebychev inequality Let X be a rv with mean $E(X) = \mu$ and variance $Var(X) = \sigma^2$. Then for any $a > 0$, we have

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Theorem 6.2

Markov inequality Let X be a positive rv. Then, for any $a > 0$

$$P(X \geq a) \leq \frac{E(X)}{a}$$