Chapter 1

Vectorial Analysis: Big Theorems

1.1 Green Theorem in the plane

Consider a "simple region" in the plane, and more precisely a region of type 1, 2 or 3.

A region of type 1: this is a region such that x is between two constants while y is between two functions of x.

A region of type 2: roles of x and y are exchanged.

A region of type 3: symmetric region.

In particular its boundary is a simple close curve: we may find an associated map $c : [a, b] \to \mathbb{R}^2$ injective over $]a, b[$, with $c(a) = c(b)$. Such a curve can be equipped with two (running) directions: the anticlockwise direction, and then we denote by C^+ this curve; and the clockwise direction, and then we denote by C^- this curve. Directions can be also associated with parts of such curves.

Let us begin with a simple lemma

Lemma 1.1.1 *Let D be a region of type 1, that is such that*

$$D = \{a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$$

C its boundary and $P : D \to \mathbb{R}$ a C^1 function. Then

$$\int_{C^+} P dx = -\int\int \partial_y p dx dy$$

The lhs is a line integral.

Proof: We shall denote by C_1 and by C_2 the "horizontal curves" and B_1 and B_2 the "vertical" curves.

As

$$D = \{a \leq x \leq b, \phi_1(x) \leq y \leq \phi_2(x)\}$$

we have

$$\int\int_D \partial_y P(x,y)dxdy = \int_a^b [\int_{\phi_1(x)}^{\phi_2(x)} \partial_y P(x,y)dy]dx =$$

$$= \int_a^b [P(x,\phi_2(x)) - P(x,\phi_1(x))]dx$$

Now, we parametrize C_1^+ by

$$x \in [a,b] \to (x, \phi_1(x))$$

and C_2^+ by

$$x \in [a,b] \to (x, \phi_2(x))$$

Then, by definition, we have

$$\int_{C_1^+} P(x,y)dx = \int_a^b P(x,\phi_1(x))dx$$

$$\int_{C_2^+} P(x,y)dx = \int_a^b P(x,\phi_2(x))dx$$

In particular

$$-\int_a^b P(x,\phi_2(x))d = \int_{C_2^-} P(x,y)dx$$

We deduce

$$\int\int_D \partial_y P(x,y)dxdy = -\int_{C_1^+} P(x,y)dx - \int_{C_2^-} P(x,y)dx$$

As x is constant on B_2^+ and B_1^-, we have

$$\int_{B_2^+} Pdx = 0 = \int_{B_1^-} Pdx$$

In conclusion, we obtain

$$\int_{C^+} Pdx = \int_{C_1^+} Pdx + \int_{B_2^+} Pdx + \int_{C_2^-} Pdx + \int_{B_1^-} Pdx$$

$$= \int_{C_1^+} Pdx + \int_{C_2^-} Pdx$$

and thus the result.

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Similarly, we have

Lemma 1.1.2 *Let D be a type 2 region, C its boundary and $Q : I\!R^2 \to I\!R$ a C^1 function. Then*

$$\int_{C^+} Q dy = \int\int_D \partial_x Q(x,y) dxdy$$

From these two results, we deduce

Theorem 1.1.1 *Green Theorem in $I\!R^2$. Let D be a type 3 region, C its boundary, and $P, Q : I\!R^2 \to I\!R$ two C^1 functions. Then*

$$\int_{C^+} Pdx + Qdy = \int\int_D [\partial_x Q(x,y) - \partial_y P(x,y)] dxdy$$

In order to apply this Theorem, a good way to remember that we are working with the positive orientation (anticlockwise ...) is to keep in mind that we travel along C so that we always keep the region D on our left.
To fix further notations, we shall always denote by ∂D what we have denoted by C^+ above. Thus Green Theorem can be formulated as

$$\int_{\partial D} Pdx + Qdy = \int\int_D [\partial_x Q(x,y) - \partial_y P(x,y)] dxdy$$

There are many applications of this result. For example

Proposition 1.1.1 *Area of a region: if C is a simple closed curve surrounding a region of $I\!R^2$ where we may apply Green Theorem, then the area of this region D is given by*

$$area(D) = \frac{1}{2}\int_{\partial D} (xdy - ydx)$$

proof: Set $P(x,y) = -y$ and $Q(x,y) = x$. Then by Green Theorem, it follows that

$$\frac{1}{2}\int_{\partial D} (xdy - ydx) = \frac{1}{2}\int_{\partial D} -ydx + xdy$$

$$= \frac{1}{2}\int\int_D [\partial_x(x) - \partial_y(-y)] dxdy$$

$$= \frac{1}{2}\int\int_D (1+1) dxdy = \int\int_D 1 dxdy = area(D)$$

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Example 1.1.1 *let us compute the area of te region enclosed by the curve C with equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$.*
One may check that we get a parametrization of C with the positive orientation, by setting

$$x = \cos^3\theta \text{ and } y = \sin^3\theta, \text{ where } 0 \le \theta \le 2\pi$$

This is called a hypocycloid. Then, we get

$$area \ = \frac{1}{2}\int_{\partial D} xdy - ydx = \ldots = \frac{3}{8}\pi$$

Theorem 1.1.2 *(vectorial form of Green Theorem in the plane). Let D be a region of $\mathbb{R}^2$ where Green Theorem applies. Denote $\partial D = C^+$. Consider that the plane $\mathbb{R}^2$ is identified with the (xOy) plane in $\mathbb{R}^3$. Let $F = (P(x,y), Q(x,y), 0)$ be a vector field over D. Then*

$$\int_{\partial D} F.ds = \int\int_D [(\ rotF).\vec{k}]dxdy$$

Example 1.1.2 *Let $F(x,y) = (xy^2, y+x)$ be a vector field of $\mathbb{R}^2$. Let D be the region of the first upper quadrant, bounded by $y = x^2$ and $y = x$. We want to compute $\int_{\partial D} F.ds$. A first way simply consists in applying the definition of a line integral. Let's apply the result just mentioned above.*
It is enough to compute $\ rotF.\vec{k}$. We find that it is equal to $1 - 2xy$. Then

$$\int_{\partial D} F.ds = \int\int_D (\ rot \ F).\vec{k}dxdy = \int_0^1 \int_{x^2}^x (1-2xy)dydx = \frac{1}{12}$$

Theorem 1.1.3 *Divergence Theorem in the plane.*
Let $D \subset \mathbb{R}^2$ be a region where Green Theorem applies, with an oriented boundary ∂D. Let $\vec{n}$ be the unit normal vector field to ∂D directed towards the exterior of D. If $c : [a,b] \to \mathbb{R}^2$, $c(t) = (x(t), y(t))$ is a parametrization of ∂D preserving its orientation, then

$$\vec{n} = \frac{(y'(t), -x'(t))}{\sqrt{(x'(t))^2 + (y'(t))^2}}$$

Let then $F = (P(x,y), Q(x,y))$ be a vector field over D. Then we have

$$\int_{\partial D} (F.n)ds = \int\int_D (divF)dxdy$$

Careful: the first integral is a line integral.

Example 1.1.3 *If $F = (y^3, x^5)$, we find that, if D is the unit square,*

$$\int_{\partial D} F.nds = \int\int div \ Fdxdy = 0$$

1.2 Stokes Theorem

Here we are going to make the link between the line integral of a vector field along a simple and closed curve C in $\mathbb{R}^3$ and a surface integral over a surface S "lying" on this curve.

To keep simple, we start first by the case where the surface $\mathcal{S}$ is associated to the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$. Let us recall then that an associated parametrization is given by

$$x = u, y = v \text{ and } z = g(u, v)$$

where $(u, v) \in D \subset \mathbb{R}^2$, and that this parametrization keeps the upwards orientation of $\mathcal{S}$. Then

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_D [F_1(-\partial_u g) + F_2(-\partial_v g) + F_3] dudv$$

where $F = (F_1, F_2, F_3)$ is a given vector field.
We assume that the set D is a region of $\mathbb{R}^2$ with a boundary ∂D being a closed and simple curve, such that we may apply 2d Green Theorem. We have also fixed an orientation of ∂D. Let $c : [a, b] \to \mathbb{R}^2$, $c(t) = (x(t), y(t))$ a positive parametrization of ∂D, thus preserving the orientation of ∂D. Then we may set

Definition 1.2.1 *With these notations, we define the oriented boundary curve $\partial\mathcal{S}$ as being the oriented simple and closed curve, obtained as the image through the map*

$$p : t \to p(t) = (x(t), y(t), g(x(t), y(t)))$$

together with the orientation induced by p.

$\partial\mathcal{S}$ has been oriented positively such as to have always $\mathcal{S}$ at the left. This orientation is said to be induced by the normal field $\vec{n}$ directed upwards (corkscrew rule).

Theorem 1.2.1 *Stokes Theorem for graphs.*
With the above notations and definition, let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field over $\mathcal{S}$. Then

$$\int\int_{\mathcal{S}^+} (\text{ } rot \text{ } F).dS = \int_{\partial\mathcal{S}^+} F.ds$$

A small remark: if G is a vector field, then

$$\int\int_{\mathcal{S}^+} G.dS = \int\int_{\Phi} G.dS = \int\int_D G.(T_u \wedge T_v) dudv =$$

$$= \int\int_D \left(\frac{T_u \wedge T_v}{\| T_u \wedge T_v \|}\right) \| T_u \wedge T_v \| \, dudv = \int\int_D (G.n) \| T_u \wedge T_v \| \, dudv = \int\int_{\mathcal{S}} (G.n) dS$$

proof: Set $F = (F_1, F_2, F_3)$. Then

$$rot\ F = (G_1, G_2, G_3) = (\partial_y F_3 - \partial_z F_2, \partial_z F_1 - \partial_x F_3, \partial_x F_2 - \partial_y F_1)$$

We deduce that

$$\int\int_{\mathcal{S}^+} (rot\ F).dS = \int\int_D (\partial_z F_1 - \partial_x F_3)(-\partial_u g) + (\partial_z F_1 - \partial_x F_3)(-\partial_v g) +$$

$$+(\partial_x F_2 - \partial_y F_1)]dudv$$

On the other hand

$$\int_{\partial\mathcal{S}} F.ds = \int_p F.ds = \int_p F_1 dx + F_2 dy + F_3 dz$$

where p denotes the path $p : [a, b] \to \mathbb{R}^3$

$$p(t) = (x(t), y(t), g(x(t), y(t))$$

Thus, applying the definition, we find

$$\int_{\partial\mathcal{S}} F.ds = \int_{\partial D} (F_1 + F_3 \partial_u g)dx + (F_2 + F_3 \partial_v g)dy$$

We apply 2d Green Theorem to get that this is equal to

$$\int\int_D [\partial_x [(F_1 + F_3 \partial_u g)] - \partial_y [(F_2 + F_3 \partial_v g)]dudv$$

and computing, we find the result.

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Example 1.2.1 *Let*

$$F = (ye^z, xe^z, xye^z)$$

We compute: rot $F = 0$ *and then we deduce that* $\int_C F.ds = 0$*, with the same notations as above.*

Example 1.2.2 *Compute*

$$\int_C -y^3 dx + x^3 dy - z^3 dz$$

where C *is the curve positively oriented, obtained as intersection of the cylinder* $x^2 + y^2 = 1$ *with the plane* $x + y + z = 1$.

Now, we want to generalize all these results to parametrized surfaces which are not necessarily obtained as graphs of functions. The key issue is to see what will be the equivalent of $\partial\mathcal{S}$ (and what is the "orientation").

Let us fix the notations. Let $\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a parametrization $\mathcal{S}$. Let $c : [a, b] \to \mathbb{R}^2$, $c(t) = (u(t), v(t))$ be a positive parametrization of ∂D.

One simple way to define ∂S would be to consider this set as a curve parametrized by

$$t \to p(t) = \Phi(u(t), v(t)))$$

This way does not work. To explain this point, let us consider an explicit example.

Let us consider a parametrization of the unit sphere $\mathcal{S}$, given as usual

$$x = \cos u \sin v, y = \sin u \sin v, z = \cos v$$

with $(u, v) \in D = [0, 2\pi] \times [0, \pi]$.

Let c be a parametrization of the boundary of the rectangle D. if we were to apply this definition of $\partial\mathcal{S}$, then we would obtain that $\partial\mathcal{S}$ would be the great circle in the plane xoz: this is not meaningful as geometrically, $\mathcal{S}$ has no boundary.

In fact this issue is connected to the fact that Φ is not injective over D. We are going to set up a restrictive definition, but which can be applied to the case of surfaces obtained as graphs of functions.

Definition 1.2.2 *With the above notations, assume moreover that Φ is injective over D. Then we call $\Phi(\partial D)$ the geometric boundary of $\mathcal{S} = \Phi(D)$. If $c(t) = (u(t), v(t))$ is a positive parametrization of ∂D, we define ∂S as the simple closed and oriented curve, obtained as image by*

$$p : t \to \Phi(u(t), v(t))$$

We say that the orientation of ∂S is induced by p.

Then we have

Theorem 1.2.2 *Stokes Theorem: parametrized surface.*

Let $\mathcal{S}$ be an oriented parametrized surface, defined by a positive and injective parametrization Φ over D. Let $\partial\mathcal{S}$ be the oriented boundary of $\mathcal{S}$. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\int\int_{\mathcal{S}^+} (rot\ F).dS = \int_{\partial\mathcal{S}} F.ds$$

Example 1.2.3 *Consider $\mathcal{S}$ as the unit upper sphere. Here ∂D is the curve with equation*

$$x^2 + y^2 = 1$$

Set $F = (y, -x, e^{xz})$. We want to compute $\int\int_{\mathcal{S}^+}(rot\ F).dS$. For this purpose, we shall use Stokes Theorem. We parametrize $\partial\mathcal{S}$ by

$$x(t) = \cos t, y(t) = \sin t, z = 0, 0 \leq t \leq 2\pi$$

Then

$$\int_{\partial\mathcal{S}} F.ds = \int_0^{2\pi} (y\frac{dx}{dt} - x\frac{dy}{dt}dt =$$

$$= \int_0^{2\pi} (-\sin^2 t - \cos^2 t)dt = \ldots = -2\pi.$$

1.3 3d Green Theorem

Let us first define what we shall term as elementary region of $\mathbb{R}^3$. Such a region of $\mathbb{R}^3$ is defined as a region such that one of the variables (say for example z) is between two functions of the other variables (here, $f(x, y) \leq z \leq g(x, y)$), and these two other variables belong to an elementary region of $\mathbb{R}^2$. A classical example is given by the unit closed ball.
Such regions are called symmetric if we can exchange the roles of these three variables: this is the case of the closed unit ball.

For such regions, the boundary can be divided into a finite number of graphs of functions.

Such (boundary) surfaces are said to be closed surfaces.

Once we have divided this surface into such small pieces, we may call face such a piece.

For usual closed surfaces, we may define two orientations: one said to be interior and the other exterior.

The choice of an orientation defines $\mathcal{S}$ as a closed oriented surface.
Then we have

Theorem 1.3.1 *3d Green Theorem. Let Ω be an elementary symmetric region of $\mathbb{R}^3$, and $\partial\Omega$ its boundary surface, closed and oriented towards the exterior. Let F be a vector field over Ω. Then*

$$\int\int\int_{\Omega}(div\ F)dxdydz = \int\int_{\partial\Omega} F.dS$$

Example 1.3.1 *Let $F = (2x, y^2, z^2)$. Let $\mathcal{S}$ be the unit sphere, oriented towards the exterior. We want to compute the flux of F accross $\mathcal{S}$, that is $\int\int_{\mathcal{S}} F.dS$. Note that $\mathcal{S}$ is a closed* *oriented surface, corresponding to $\partial\Omega$, if Ω denotes the unit closed ball, which is an elementary symmetric region of $I\!R^3$. Then, applying 3d Green Theorem, we have*

$$\int\int_{\mathcal{S}} F.dS = \int\int\int_{\Omega} (div\ F) dxdydz = \ldots = 8\frac{\pi}{3}$$

1.4 Exercices of this Chapter

2d Green

1. Compute $\int_C y\ dx - x\ dy$ where C is the boundary of the square $[-1,1]\times[-1,1]$, oriented positively.

2. Compte the area of a disk of radius R using 2d Green Theorem.

3. Check Green theorem for the disk D centered at $(0,0)$ and with radius R:

   (a) $P(x,y) = xy^2,\ Q(x,y) = -yx^2$,

   (b) $P(x,y) = x+y,\ Q(x,y) = y$,

   (c) $P(x,y) = xy = Q(x,y)$,

   (d) $P(x,y) = 2y,\ Q(x,y) = x$.

4. Under the conditions of Green Theorem, show that

   (a)
   $$\int_{\partial D} PQ\ dx + PQ\ dy = \int\int_D [Q(\partial_x P - \partial_y P) + P(\partial_x Q - \partial_y Q)]\ dxdy.$$

   (b)
   $$\int_{\partial D} (Q\partial_x P - P\partial_x Q)\ dx + (P\partial_y Q - Q\partial_y P)\ dy = 2\int\int_D (P\partial^2_{xy} Q - Q\partial^2_{xy} P)\ dxdy$$

5. Compute $\int_C (2x^3 - y^3)dx + (x^3 + y^3)dy$, where C is the unit circle.

6. (a) Check Divergence Theorem for $F = (x,y)$ and the unit disk D.

   (b) Compute the integral of the normal component of $(2xy, -y^2)$ along the ellipsis defined by $x^2/a^2 + y^2/b^2 = 1$.

Stokes

7. Check Stoke Theorem for $z = \sqrt{1 - x^2 - y^2}$, $z \geq 0$ and the field $F = (x,y,z)$.

8. Let S the surface defined by $S = S_1 \cup S_2$, where S_1 is the surface $x^2+y^2 = 1$, $0 \leq z \leq 1$ and S_2 is the surface $x^2+y^2+(z-1)^2 = 1$, $z \geq 1$. Let $F = (zx+z^2y+x, z^3yx+y, z^4x^2)$. Compute $\int\int_S (rot\ F).dS$.

9. Compute $\int\int_S (rot\ F).dS$, where S is the surface defined by $x^2+y^2+z^2 = 1$, $x+y+z \geq 1$ and $F = (x, y, z)$.

10. Compute $\int\int_S (rot\ F).dS$, where S is $x^2 + y^2 + z^2 = 1$, $x \geq 0$, and $F = (x^3, -y^3, 0)$.

11. Compute $\int\int_S (rot\ F).dS$, where S is $x^2 + y^2 + 2z^2 = 10$, et $F = (\sin(xy), e^x, -yz)$.

3d Green Theorem

12. Let S be a closed surface. Let F be a vector field. Show that $\int\int_S (rot\ F).dS = 0$.

13. Let $F = (x^3, y^3, z^3)$. Compute the flux of F accross the unit sphere.

14. Let Ω be the unit cube (in the usual part of $\mathbb{R}^3$). Compute $\int\int_{\partial\Omega} F.dS$ (two ways):

    (a) $F = (x, y, z)$

    (b) $F = (1, 1, 1)$

    (c) $F = (x^2, x^2, z^2)$

15. Let $F = (x, y, xz)$. Compute $\int\int_{\partial\Omega} F.dS$ when Ω is given by

    (a) $x^2 + y^2 \leq z \leq 1$

    (b) $x^2 + y^2 \leq z \leq 1$ and $x \geq 0$

    (c) $x^2 + y^2 \leq z \leq 1$ ad $x \leq 0$

16. Same as the previous exercice with $F = (x - y, y - z, z - x)$.