Chapter 4

Fourier series

General idea: represent a complicated function as a sum of very simple functions.
Here we take as simple functions: cos and sin.

4.1 Approximation by infinite sums

Let $f : [a, b] \to \mathbb{R}$ be given. We want to replace f by a more simple function.
For any integer $n \geq 0$, let be given functions $\phi_n : [a, b] \to \mathbb{R}$. These functions are supposed to be simpler than f. In practise, these functions are polynomials, or linear combinations of cos and sin type functions.
We then want to approximate f by a finite linear combination of these functions ϕ_i. That is, we want to approximate f by a sum s_N, where N is an intgeer, with

$$s_N(x) = \sum_{i=1}^{N} c_i \phi_i(x)$$

The coefficients c_i should be in fact found such that this sum s_N will be close to f. The meaning of "close" will be detailed below. The most simple way is given by

Définition 4.1.1 *Let be given coefficients c_i and fonctions $\phi_i : [a, b] \to \mathbb{R}$ for all integer i. We say that the series $\sum_{n\geq 1} c_n \phi_n$ converges simply towards the function f if for any $x \in [a, b]$ fixed, the numerical series $\sum_{n\geq 1} c_n \phi_n(x)$ converges and has a sum equal to $f(x)$. Thus, if we set for any integer N and for all $x \in [a, b]$*

$$s_N(x) = \sum_{n=1}^{N} c_n \phi_n(x)$$

we should have

$$\lim_{N\to+\infty} s_N(x) = f(x)$$

In that case, we shall write $\sum_{n=1}^{+\infty} c_n\phi_n = f$.

Another convergence is given by quadratic (mean) convergence

Définition 4.1.2 *With the same notations, we say that the series* $\sum_{n\geq 1} c_n\phi_n$ *converges quadratically towards the function* f *if we have*

$$\lim_{N\to+\infty} \int_a^b \mid f(x) - s_N(x) \mid^2 dx = 0$$

Définition 4.1.3 *Let* ϕ *et* ψ *be two functions* $[a,b] \to I\!R$. *We say that they are orthogonal if we have*

$$\int_a^b \psi(x)\phi(x)dx = 0$$

For example: $a = 1$, $b = -1$, $\psi = 1$ et $\phi = x$.

Définition 4.1.4 *Let be given for all integer* n, *a function* $\phi_n : [a,b] \to I\!R$. *We say that they are orthogonal if for all different integers* n *and* m, *we have*

$$\int_a^b \phi_n(x)\phi_m(x)dx = 0$$

We also say that the sequence $\{\phi_n\}$ *is orthogonal*

Now, our goal is to find the coefficients c_n such that the series $\sum_{n\geq 0} c_n\phi_n$ converges quadratically towards f, with the assumption that the sequence $\{\phi_n\}$ is orthogonal. Starting from

$$\int_a^b \mid f(x) - s_N(x) \mid^2 dx$$

we expand to get

$$\int_a^b \mid f(x) - \sum_{n=1}^{N} c_n\phi_n(x) \mid^2 dx = \int_a^b fr(x)dx - 2\sum_{n=1}^{N} c_n \int_a^b f\phi_n dx + \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2 dx$$

The r.h.s can be written as

$$\sum_{n=1}^{N} \int_a^b \phi_n^2 \{c_n - \frac{\int_a^b f\phi_n}{\int_a^b \phi_n^2}\}^2 + \int_a^b f^2 - \sum_{n=1}^{N} \frac{[\int_a^b f\phi_n]^2}{\int_a^b \phi_n^2}$$

Here we need to assume that

$$\int_a^b \phi_n^2(x)dx \neq 0 \text{ for all } n$$

The coefficients c_n only appear in the first term. As it is a sum of squares (positive), if we want this to be very small, it is enough to choose c_n such that this term is zero. That is we make the choice

$$c_n = \frac{\int_a^b f\phi_n}{\int_a^b \phi_n^2}$$

Définition 4.1.5 *With the previous assumptions, set*

$$c_n = \frac{\int_a^b f\phi_n}{\int_a^b \phi_n^2}$$

Then these coefficients are called the Fourier coefficients of f wr.t. the orthogonal sequence $\{\phi_n\}$. We then say that the series $\sum_{n\geq 1} c_n\phi_n$ is the Fourier series of f w.r.t. the sequence $\{\phi_n\}$.

We use the notation

$$f(x) \simeq \sum_{n\geq 1} c_n\phi_n$$

to say that the series on the r.h.s. is the Fourier series of f.

Caution: we do not know if this series converges, simply or quadratically.

4.2 Completion

We have seen that

$$\int_a^b \mid f(x) - \sum_{n=1}^{N} c_n\phi_n(x) \mid^2 dx = \int_a^b f^2(x)dx - \sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2(x)dx$$

We then deduce

Proposition 4.2.1 *Bessel inequality. With the previous assumptions, let c_n be the Fourier coefficients of f w.r.t. the orthogonal sequence $\{\phi_n\}$. Then we have the inequality*

$$\sum_{n=1}^{N} c_n^2 \int_a^b \phi_n^2(x)dx \leq \int_a^b f^2(x)dx$$

In fact we can even deduce that

Proposition 4.2.2 *Generalized Bessel inequality. With the same assumptions, the numerical and positive series $\sum_{n\geq 1} c_n^2 \int_a^b \phi_n^2(x)dx$ converges and moreover its sum satisfies*

$$\sum_{n=1}^{+\infty} c_n^2 \int_a^b \phi_n^2(x)dx \leq \int_a^b f^2(x)dx$$

Remember that we should also have

$$c_n^2 \int_a^b \phi_n^2(x)dx \to 0 \text{ when } n \to +\infty$$

Finally we have

Proposition 4.2.3 *If the Fourier series of f w.r.t. the orthogonal sequence $\{\phi_n\}$ converges quadratically towards f, then we have*

$$\int_a^b f^2(x)dx = \sum_{n=1}^{+\infty} c_n^2 \int_a^b \phi_n^2(x)dx$$

The opposite is also true.

This is Parseval equality.

4.3 Classical Fourier series

We choose the interval $(-\pi, \pi)$ and functions ϕ_n:

$$1, \ , \cos x, \sin x, \cos 2x, \sin 2x, \ldots$$

These functions are continuous by pieces and their squares are integrable. Moreover we have

$$\int_{-\pi}^{\pi} 1^2 dx = 2\pi$$

$$\int_{-\pi}^{\pi} \cos^2 nx dx = \int_{-\pi}^{\pi} \sin^2 nx dx = \pi, n = 1, 2, \ldots$$

Let us set

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt dt, n = 0, 1, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt dt, n = 1, \ldots$$

Then the Fourier series of f is given by

$$f(x) \simeq \frac{1}{2} a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots$$

$$\simeq \frac{1}{2} a_0 + \sum_{n \geq 1} a_n \cos nx + b_n \sin nx$$

Définition 4.3.1 *Let f be continuous by pieces and square integrable. Then its Fourier coefficients are the numbers*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt, n = 0, 1, \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt, n = 1, \dots$$

The Fourier series of f is:

$$f(x) \simeq \frac{1}{2}a_0 + \sum_{n\geq 1} a_n \cos nx + b_n \sin nx$$

One can check that for all distinct n and m, we have $\int_{-\pi}^{\pi}\cos nx \sin mx dx = 0$ (orthogonality. Here are some sufficient conditions to check the simple convergence of the Fourier series towards f.

Proposition 4.3.1 *The Fourier series of f at a given x converges towards $f(x)$, that is we have the equality*

$$f(x) = \sum_{n\geq 1} a_n \cos nx + b_n \sin nx$$

under one of the following conditions:

1) f is continuous by pieces and integrable and that f is derivable at x fixé.

2) f is Lipschitz of order α at x that is there exists two constants M and α such that

$$\mid f(y) - f(x) \mid \leq M \mid y - x \mid^{\alpha}, \ \textit{for all } y \in (-\pi, \pi)$$

Caution: it may appear that if f is only continuous at x, then its Fourier series diverges at x.

Assume now that f is discontinuous at x but have limits $f(x+0)$ and $f(x-0)$.

Proposition 4.3.2 *Assume one of the following conditions at a fixed point x:*

1) f is continuous by pieces and integrable and derivable around a small interval cetered at x and that this derivative is bounded there (maybe except at x).

2) f is continuous by pieces and integrable and Holder at x, that is there exists two positive constants M and α such that

$$\mid f(x) - f(y) \mid \leq M \mid y - x \mid^{\alpha} \quad \forall y \in [-\pi, \pi]$$

Then we have

$$\lim_{N\to+\infty} s_N(x) = \frac{1}{2}[f(x+0) + f(x-0)]$$

that is the Fourier series of f at x converges to the mean $\frac{1}{2}[f(x+0) + f(x-0)]$.

4.4 Other types of convergence

Définition 4.4.1 *We say that the Fourier series of f converges uniformly towards a function g if for all fixed ε f(the error), we can choose N such that $\mid f(x) - s_N(x) \mid < \varepsilon$, and this for all x.*

Caution: Uniform convergence towards $g = f$ implies simple convergence (but the opposite is not true).
If f is discontinuous (with a jump) at a given point, there cannot be uniform convergence of its Fourier series.

In fact we have

Proposition 4.4.1 *If a Fourier series of a function f converges uniformly towards f, then necessarily this function f should be continous and should satisfy , that is f should be 2π periodic.*

Proposition 4.4.2 *Let f be a continuous and 2π periodic function. Assume that f is derivable over $[-\pi, \pi]$ out of a finite number of points. Assume that out these points, this derivative is continuous and that the integral $\int_{-\pi}^{\pi} f'^2(x)dx$ converges. Then we have*

$$f(x) \simeq \frac{1}{2}a_0 + \sum_1^\infty (a_n \cos nx + b_n \sin nx)$$

et

$$f'(x) \simeq \sum_1^\infty (nb_n \cos nx - na_n \sin nx)$$

Théorème 4.4.1 *Let f be a continuous function, 2π-periodic and such that $\int_{-\pi}^{\pi} f'^2 dx$ is finite. Then its Fourier series convers uniformly towards f.*

Exemple 4.4.1 *Let $f(x) = \mid x \mid$ for all $x \in [-\pi, \pi]$ and 2π-periodic. Then we have*

$$f(x) \simeq \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^\infty \frac{\cos(2k-1)x}{(2k-1)^2}$$

Note that

$$f'(x) = -1 \text{ if } -\pi < x < 0 \text{ and } = 1 \text{ if } 0 < x < \pi$$

and thus f' is discontinuous at a finite number of points in $[-\pi; \pi]$. Note that f' is not defined at 0. Finally $\int_{-\pi}^{\pi} f'^2 dx$ is finite.

Proposition 4.4.3 *If the Fourier series of a function f converges uniformly towards f, the is also converges quadratically.*

In fact we have also

Proposition 4.4.4 *Let f be a continous by pieces function on $[-\pi;\pi]$, $2\pi-$ periodic and such that $\int_{-\pi}^{\pi} f^2 dx$ is finite. Then its Fourier series converges quadratically towards f.*

We have also Parseval relation

Proposition 4.4.5 *Let f and f^* two continuous by pieces functions with $\int_{-\pi}^{\pi} f^2(x)dx < +\infty$ and $\int_{-\pi}^{\pi} f^{*2}(x)dx < +\infty$. Then if*

$$f(x) \simeq \frac{1}{2}a_0 + \sum_1^\infty [a_n \cos nx + b_n \sin nx],$$

$$f^*(x) \simeq \frac{1}{2}a_0^* + \sum_1^\infty [a_n^* \cos nx + b_n^* \sin nx],$$

we have the Pareseval equality

$$\int_{-\pi}^{\pi} f(x)f^*(x)dx = \pi[\frac{1}{2}a_0 a_0^* + \sum_1^\infty (a_n a_n^* + b_n b_n^*)]$$

Exemple 4.4.2 *Let us take the 2π periodic functions defined on $[-\pi,\pi]$ by*

$$f(x) = x \ \ et \ f^*(x) = x^3$$

Firstly we have the Fourier expansions

$$f(x) \simeq -2\sum_1^\infty \frac{(-1)^n}{n} \sin nx$$

$$f^*(x) = -2\sum_1^\infty (\frac{\pi^2}{n} - \frac{6}{n^3})(-1)^n \sin nx$$

Using Parseval equality, we deduce that

$$\frac{2\pi^5}{5} = \int_{-\pi}^{\pi} x^4 dx = 4\pi \sum_1^\infty \frac{1}{n^2}(\pi^2 - \frac{6}{n^2})$$

Then, let us note that if f is continuous and 2π periodic, with $\int f'^2 dx < \infty$, then we can apply Parseval inequlity to f' to get

$$\sum_1^\infty n^2(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'^2 dx$$

We deduce that for all M,

$$\sum_{M+1}^{\infty} n^2(a_n^2+b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'^2 dx - \sum_{1}^{M} n^2(a_n^2+b_n^2)$$

If we apply Parseval inequality to $f(x)=x$, we get

$$\sum_{1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

Letting $N \to +\infty$ in the above equality linking the difference between s_N and s_M, we obtain an error made if we approximate f by its Fourier partial sum s_M

$$\mid f(x) - s_M(x) \mid \leq \{\frac{1}{\pi}\int_{-\pi}^{\pi} f'^2 dx - \sum_{1}^{M} n^2(a_n^2+b_n^2)\}^{\frac{1}{2}}\{\frac{\pi^2}{6} - \sum_{1}^{M}\frac{1}{n^2}\}^{\frac{1}{2}}$$

Exemple 4.4.3 *Let $f(x) = \mid x \mid$ on $[-\pi,\pi]$. Then*

$$\int_{-\pi}^{\pi} f'^2(x)dx = 2\pi$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{1}^{\infty}\frac{\cos(2k-1)x}{(2k-1)^2}$$

Taking $M=2$, we get (note that $a_2=b_2=0$)

$$\mid\mid x \mid - [\frac{\pi}{2} - \frac{4}{\pi}\cos x] \mid \leq \{2 - \frac{16}{\pi^2}\}^{\frac{1}{2}}\{\frac{\pi^2}{6} - 1 - \frac{1}{4}\}^{\frac{1}{2}} = 0.39$$

In fact, the maximum is obtained at $x=0$ and at $x=\mp\pi$, and is equal to 0.30.

4.5 Cosinus and Sinus series

If f is an odd function, then all coefficients a_n are zero. In that case, we have

$$f(x) \simeq \sum_{1}^{\infty} b_n \sin nx$$

and moreover

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

If f is an even function, then all coefficients b_n are zero. In that case, we have

$$f(x) \simeq \frac{a_0}{2} + \sum_{1}^{\infty} a_n \sin nx$$

and moreover

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx dx$$

4.6 Change of scales

We have seen that the sequence $\{\cos nx, \sin nx\}$ was orthogonal over the interval $[-\pi, \pi]$. This fact is well adapted for the decomposition of functions defined on this interval or even on half of this interval if we take into consideration the parity of this function.
We can in fact proceed similarly for functions defined over an arbitrary interval $[a, b]$. For this purpose, let us introduce the new variable

$$\bar{x} = \frac{2\pi(x - \frac{1}{2}(a+b))}{b-a}$$

for any $x \in [a, b]$. This new coordinate $\bar{x}$ sends $x \in [a, b]$ on $\bar{x} \in [-\pi, \pi]$. Also, one can check that

$$x = \frac{b-a}{2\pi}\bar{x} + \frac{1}{2}(a+b)$$

The interest of this new variable is that we are back to the interval $[-\pi, \pi]$. Now for any variable $\bar{x}$ in $[-\pi, \pi]$, introduce the function F defined on $[-\pi, \pi]$ by

$$F(\bar{x}) = f(\frac{b-a}{2\pi}\bar{x} + \frac{1}{2}(a+b))$$

Note that $F(\bar{x}) = f(x)$. The function F is well defined over $[-\pi, \pi]$, and we may consider its usual Fourier series

$$F(\bar{x}) \simeq \frac{1}{2} + \sum_{1}^{\infty}(a_n \cos n\bar{x} + b_n \sin n\bar{x})$$

where

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\bar{x}) \cos n\bar{x} d\bar{x}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(\bar{x}) \sin n\bar{x} d\bar{x}$$

This Fourier series converges quadratically towards F if $\int_{-\pi}^{\pi} F(\bar{x})^2 d\bar{x}$ is finite, uniformly if F is continuous, 2π periodic and $\int_{-\pi}^{\pi} F'^2 d\bar{x}$ is finite.
Now if we remember the link between the variables x and $\bar{x}$, we find that

$$f(x) \simeq \frac{1}{2}a_0 + \sum_{1}^{\infty}[a_n \cos\frac{2\pi n}{b-a}(x - \frac{1}{2}(a+b)) + b_n \sin\frac{2\pi n}{b-a}(x - \frac{1}{2}(a+b))],$$

with

$$a_n = \frac{2}{b-a}\int_a^b f(x) \cos\frac{2\pi n}{b-a}(x - \frac{1}{2}(a+b))dx$$

$$b_n = \frac{2}{b-a}\int_a^b f(x) \sin\frac{2\pi n}{b-a}(x - \frac{1}{2}(a+b))dx$$

Note that the family $\{\cos\frac{2\pi n}{b-a}(x-\frac{1}{2}(a+b)), \sin\frac{2\pi n}{b-a}(x-\frac{1}{2}(a+b))\}$ is orthogonal over the interval $[a,b]$.

We see immediately that this Fourier series converges quadratically towards f if $\int_a^b f(x)^2dx$ is finite, and uniformly if f is continuous, with $f(a)=f(b)$ and $\int_a^b f'^2(x)dx$ is finite. All convergence theorems above remain true here. Similarly, according to the parity of f, we may expand f in Sin series

$$f(x) \simeq \sum_1^\infty b_n \sin\frac{\pi n}{b-a}(x-a)$$

with

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin\frac{\pi n}{b-a}(x-a)dx$$

or in Cos series

$$f(x) \simeq \frac{1}{2} + \sum_1^\infty a_n \cos\frac{\pi n}{b-a}(x-a)$$

with

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos\frac{\pi n}{b-a}(x-a)dx$$

4.7 Fourier series with multi variables

Let us deals to simplify with the 2d case.

More precisely, consider a function $f : \mathbb{R}^2 \to \mathbb{R}$, the variables here being the couple $(x,y)\in\mathbb{R}^2$ and we assume that f is C^1 over $\mathbb{R}^2$ and 2π periodic w.r.t. each of these variables that is

$$f(x+2\pi,y) = f(x,y+2\pi) = f(x,y)$$

for all $(x,y)\in\mathbb{R}^2$. In particular, if we know f over a square (with sides parallel to the coordinate axis) of length 2π. In practise, we'll take the square $[-\pi,\pi]\times[-\pi,\pi]$.

Let us fix the variable y for example. Then we may consider the function f as a function of the single variable x. Then its Fourier series (w.r.t. variable x) is uniformly convergent (towards $f(x,y)$) that is we have

$$f(x,y) = \frac{1}{2}a_0 + \sum_1^\infty [a_n\cos nx + b_n \sin nx]$$

and this holds for all $x\in\mathbb{R}$.

In fact, as we have fixed the variable x, coefficients a_n and b_n also depend on y and thus we have

$$f(x,y) = \frac{1}{2}a_0(y) + \sum_1^\infty [a_n(y)\cos nx + b_n(y)\sin nx]$$

Thus coefficients a_n and b_n are also function of variable y and are given by

$$a_n(y) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx dx$$

$$b_n(y) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx dx$$

Then for all n, we have

$$a_n(y) = \frac{1}{2}a_{n0} + \sum_{1}^{\infty}(a_{nm}\cos my + b_{nm}\sin my)$$

$$b_n(y) = \frac{1}{2}c_{n0} + \sum_{1}^{\infty}(c_{nm}\cos my + d_{nm}\sin my)$$

with

$$a_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx\cos my dx dy$$

$$b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\cos nx\sin my dx dy$$

$$c_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx\cos my dx dy$$

$$d_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\sin nx\sin my dx dy$$

On the whole, we get

$$f(x,y) = \frac{1}{4}a_{00} + \frac{1}{2}\sum_{m=1}^{\infty}[a_{0m}\cos my + b_{0m}\sin my] + \frac{1}{2}\sum_{n=1}^{\infty}[a_{n0}\cos nx + c_{n0}\sin nx] +$$

$$+\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm}\cos nx\cos my + b_{nm}\cos nx\sin my + c_{nm}\sin nx\cos my + d_{nm}\sin nx\sin ny]$$

Using Parseval equality, we get

$$\int_{-\pi}^{\pi} f(x,y)^2 dx = \frac{\pi}{2}a_0(y)^2 + \pi\sum_{n=1}^{\infty}[a_n(y)^2 + b_n(y)^2]$$

and integrating, we get

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)^2 dx dy = \frac{\pi^2}{2}\int_{-\pi}^{\pi} a_0^2 dy + \pi^2\int_{-\pi}^{\pi}[a_n^2 + b_n^2]dy$$

We have also

$$\int_{-\pi}^{\pi} a_n(y)^2 dy = \frac{\pi}{2}a_{n0}^2 + \pi\sum_{m=1}^{\infty}(a_{nm}^2 + b_{nm}^2)$$

$$\int_{-\pi}^{\pi} b_n(y)^2 dy = \frac{\pi}{2} c_{n0}^2 + \pi \sum_{m=1}^{\infty} (c_{nm}^2 + d_{nm}^2)$$

In conclusion, we get

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y)^2 dx dy = \frac{\pi^2}{4} a_{00}^2 + \frac{\pi^2}{2} \sum_{n=1}^{\infty} (a_{n0}^2 + c_{n0}^2) +$$

$$+\pi^2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (a_{nm}^2 + b_{nm}^2 + c_{nm}^2 + d_{nm}^2)$$

This is called the Pareseval equality and it holds true for function continuous by pieces and such that $\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y)^2 dx dy$ est finie.

4.8 Exercices of this Chapter

1. Let $f : \mathbb{R} \to \mathbb{R}$, odd, 2π periodic, given by

$$f(t) = t \text{ if } 0 \leq t < \frac{\pi}{2},\ f(t) = \pi - t \text{ if } \frac{\pi}{2} \leq t \leq \pi.$$

   (a) Check that f is continuous by pieces and compute its Fourier coefficients.

   (b) Study the convergence of its Fourier series.

   (c) Deduce the sum of the following series:

$$\sum_{p=0}^{\infty} \frac{1}{(2p+1)^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{p=0}^{\infty} \frac{1}{(2p+1)^4}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

2. Give the Fourier series of f, 2π periodic $f(x) = x^2$ sur $]-\pi; +\pi]$. Deduce the sum of the following two series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

3. Compute the Fourier series of f, 2π periodic such that $f(x) = x^2 - \pi^2$ on $]-\pi; +\pi]$. Compute $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

4. Let$f : \mathbb{R} \to \mathbb{R}$, 2π periodic, defined for $x \in [-\pi, \pi]$ by $f(x) = \cosh x$.

   (a) Compute its Fourier series

(b) Deduce the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2+1}, \qquad \sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2}.$$

(c) What may we deduce for the function g, 2π periodic, such that $g(x) = \sinh x$ for $x \in [-\pi, \pi]$?

5. Let $x \in]0, \frac{\pi}{2}]$. We define the continuous function f_x, 2π periodic on $\mathbb{R}$, even and linear by pieces, by $f_x(0) = 1$ and $f_{x|[2x,\pi]} = 0$.

   (a) Compute the Fourier series of f_x.

   (b) Deduce the sum of the following series:

$$\sum_{k=1}^{\infty} \frac{\sin^2 kx}{k^2}, \qquad \sum_{k=1}^{\infty} \frac{\sin^4 kx}{k^4}.$$

6. Show that for all $x \in]0, 2\pi[$, we have

$$\frac{\pi}{8}(\pi - x) = \sum_{n=0}^{\infty} \frac{\cos(n+\frac{1}{2})\, x}{(2n+1)^2}.$$

7. Show that $| \sin x | = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin^2 nx}{4n^2-1}$.

8. Compute the Fourier series of the 2π periodic function f defined for $x \in [0, 2\pi[$ by $f(x) = e^{ax}$, a being a fixed real number.

   Deduce the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$.

9. Compute the Fourier series of the odd and 2π periodic function f such that $f(x) = x(\pi - x)$ sur $[0, \pi]$. Deduce the sum of the following series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}, \qquad \sum_{n=0}^{\infty} \frac{1}{(2n+1)^6}.$$

10. Compute the Fourier series of the 2π periodic functon f such that

$$f(x) = 0 \text{ if } -\pi < x < 0,\ f(x) = x^2 \text{ if } 0 < x < \pi$$

   Deduce the sum of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$