Probability and Statistics
Lecture 1
Probability Spaces

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Contents

Generalities

A random phenomena, reproduced a number of times with exactly the same conditions, will give different results.
These results are different from each other in a unpredictable fashion.
Examples : flipping a coin (flipping heads or tails), dice rolling, lifetime of a light bulb, time arrival of a sailing boat ...

A random experiment $\mathcal{E}$: repeated within the same conditions, can lead to different possible results (outcomes).
The set of all possible outcomes (samples) is usually called :
the state space, the realization space,
the space of events, the space of samples.

Denoted by Ω.

A possible outcome $\omega \in \Omega$:basic or simple event.
An event A is a subset of the full space of events Ω : $A \subset \Omega$.
$\emptyset$: impossible event ; Ω : certain event.
If Ω is a finite set with *card* $\Omega = n$, then there are 2^n events (simple).

Example

throwing two coins head/tail : $\Omega = \{HH, TT, TH, HT\}$. $\odot$

Example

rolling a dice $\Omega = \{1, 2, 3, 4, 5, 6\}$.$\odot$

Example

throwing a dart on a circular target of 30cm diameter. experiment : describe the impact of the dart in a orthonormal frame centered at the center of the target : : $\Omega = \{(x, y), x^2 + y^2 \leq 15^2\}$. $\odot$

Example

Lifetime of a light bulb : $\Omega = [0, +\infty[$. $\odot$

A random experiment : $->$ one and only one basic event..
Basic events are incompatible : that is two basic events cannot occur at the same time. Morover basic events are exhaustive.
Ω is exactly equal to the union of all basic events.
If Ω is finite or infinite countable : discrete.
IfΩ is infinite non countable : it is said continuous.

Example

Dice (6 faces) rolling.
a) A possible outcome, for ex. 5 : one realization, ω.
b) All possible outcomes $\{1,2,3,4,5,6\}$: set of realizations or state space, Ω.
c) An event throw : an example of an event, $A \subset \Omega$. The opposite event : A^c.
d) Certain event :Ω ;
e) Impossible event : $\emptyset$;
f) The event A and B is $A \cap B$;
g) The event A or B (non exclusive) is $A \cup B$;
h) A and B are incompatible if $A \cap B = \emptyset$.
$\odot$

Probability Notion

Example of a dice rolling (with 6 faces).
Question : probability (that is the chance) to get "2" ?
Test : a large number N of identical throws and then count the number of "2" out of these throws.
A : event (basic) get "2" ;
N(A) : number of obtained "2".
The empirical (statistical, sampling) frequency of success $\frac{N(A)}{N}$ should be close to 1/6 : $P(A)$, that is the probability of getting A.
This is so if the die is not rigged.
By using a large set of throws, it should appear that

$$P(\Omega) = 1, \; P(\emptyset) = 0,$$

and for two incompatible events, that is such that $A \cap B = \emptyset$, we have

$$P(A \cup B) = P(A) + P(B).$$

Therefore, we have a map from $\mathcal{P}(\Omega)$ into $\mathbb{R}$.

We set

$$\mathcal{T} = \mathcal{P}(\Omega)$$

Definition 3.1

A probability (on Ω) is a map (application) P from $\mathcal{T}$ into $[0,1]$ such that :
- $P(\Omega) = 1$ and $P(\emptyset) = 0$.
- If A_i, $i \in I$, is an at most countable set of 2 by 2 disjoint events, then (σ-additivity)

$$P(\cup_{i\in I} A_i) = \sum_{i\in I} P(A_i)$$

$(\Omega, \mathcal{T}, P)$: a probability space.

An event A is said almost certain if $P(A) = 1$ and negligeable if $P(A) = 0$.

Example

Throw of a die : $\Omega = \{1,2,3,4,5,6\}$, $\mathcal{T} = \mathcal{P}(\Omega)$ and $P(\{\omega\}) = P(\omega) = 1/6$, for all $\omega \in \Omega$. $\odot$

Proposition 3.1

For any event A and B in $\mathcal{T}$, we have :

1. $P(A^c) = 1 - P(A)$;
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
3. $P(A) \leq P(B)$ *si* $A \subset B$;
4. *If A_i, $i \in I$, is an at most countable set of 2 by 2 disjoint events, and covering Ω, then*

$$P(B) = \sum_{i \in I} P(A_i \cap B)$$

Proof 1

exercice.

The case finite or countable

If Ω is at most countable, then we take $\mathcal{T} = \mathcal{P}(\Omega)$.
$->$ the probabiliy$P$ is determined by the values $P(\omega)$, for all $\omega \in \Omega$.
$->$ for all event $A \in \mathcal{T}$, we have

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Definition 4.1

If Ω is finite, we call ***uniform probability*** *on Ω the probability P defined by*

$$P(\omega) = \frac{1}{\text{card } \Omega}, \text{ for all } \omega \in \Omega$$

In this case, for any event A, we have

$$P(A) = \frac{\text{card } A}{\text{card } \Omega}$$

Counting/Enumeration

Facts :

- number of permutations (or bijections) of $\{1, ..., n\}$ is $n!$.

- number of arrangements of k elements from n, or number of injections from $\{1, ..., k\}$ into $\{1, ..., n\}$ is $A_n^k = \frac{n!}{(n-k)!}$.

- number of subsets with k elements from a set with n elements is $C_n^k = \frac{n!}{k!(n-k)!}$.

- Binomial formulae :

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

Draws and Urns

Let N balls with k different colors : N_1 balls with color 1, ..., N_k balls with color k.
Fraction of balls with color i is

$$p_i = N_i / N.$$

Experiment : we draw randomly n balls from the urn, with $n \leq N$.
Problem : repartition (distribution) of obtained colors and computation of

$$P_{n_1 n_2 \ldots n_k}$$

which is the probability to obtain n_1 balls with color 1..., with $n_1 + n_2 + \ldots + n_k = n$.
Defnition of a draw : draw with reset, draw without reset, simultaneous draw.

a) Simultaneous draw

We draw all balls in the same draw.
Ω : set of all possible sets of length n of different elements among N : number C_N^n.

$$P_{n_1 n_2 \dots n_k} = \frac{C_{N_1}^{n_1} \dots C_{N_k}^{n_k}}{C_N^n}$$

polygeometric distribution .
Specific case of two colors :

$$P_{n_1, n-n_1} = \frac{C_{N_1}^{n_1} C_{N-N_1}^{n-n_1}}{C_N^n}$$

is the hypergeometric law .

b) Draw with reset

Successive draws, with reset the drawn ball at each time.
Ω : set of all the n-couples of elements of the urn. *Card* $\Omega = N^n$.
Unform probability on Ω.

$$P_{n_1 \dots n_k} = \frac{n!}{n_1! n! \dots n_k!} \frac{N_1^{n_1} \dots N_k^{n_k}}{N^n}.$$

multinomial distribution. If $k = 2$, $p_1 = p$ and $p_2 = 1 - p$,

$$P_{n_1, n-n_1} = C_n^{n_1} p^{n_1} (1-p)^{n-n_1}.$$

which is the binomial law with parameters n and p.

c) Draw without reset

We draw the balls sucessively, but without reset. Ω is the set of sequences of n different elements among N, the number of which is A_N^n.
One can show that we obtain the same probability as in the simultaneous case.
Thus equivalence between draw without reset and simultaneous draw .

Geometric probabilities

Let Ω be a bounded regular [what is it] set of $\mathbb{R}^n$. We set :

$$P(A) = \frac{|A|}{|\Omega|}$$

$|A|$ and $|\Omega|$: measure of these sets, that is lenght, area or volume in dimension 1, 2 or 3 resp.
We construct $\mathcal{T}$ by at most countable intersection or union of cubes [see lectures on integratino theory]

Example

$\Omega = [a, b] \subset \mathbb{R}$; we take

$$P([c, d]) = \frac{d - c}{b - a}.$$

$\odot$

Conditional Probabilities

Definition 6.1

Let A and B two events, with $P(B) \neq 0$. We call ***conditional probability*** *of A w.r.t B or relative to B or knowing B, the number*

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}.$$

We have

$$P(A^c|B) = 1 - P(A|B).$$

Proposition 6.1

Multiplication Rule *If $P(B) \neq 0$, then*

$$P(A \cap B) = P(A|B)P(B) \text{ and } P(A \cap B) = P(B|A)P(A) \text{ if } P(A) \neq 0$$

Proposition 6.2

Bayes formulae *If* $P(A).P(B) \neq 0$, *then*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Definition 6.2

Partition *Let* B_1, B_2, ..., B_n *be a sequence (at most countable) of 2 by 2 disjoint events and whose union covers the state of events* Ω. *Then we say that this sequence is a partition of* Ω.

Assumption : it is assumed that these events are of strictly positive probabilities. The most simple example is given by A and A^c.

Proposition 6.3

Total probability rule *If $(B_i)_{i \in \mathbb{N}}$ is a partition of Ω, then for all $A \subset \Omega$, we have*

$$P(A) = \sum_{k \in \mathbb{N}} P(A \cap B_k) = \sum_{k \in \mathbb{N}} P(A|B_k)P(B_k)$$

if $P(B_k) > 0$ for any k.

Finally, applying the previous results, we have

Proposition 6.4

Bayes Rule

Let $(B_i)_i$ be a partition of Ω of non null probability. Then for any $A \subset \Omega$, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_k P(A|B_k)P(B_k)} \text{ for all } j$$

Definition 6.3

Two events A and B are called *independent* *iff*

$$P(A \cap B) = P(A)P(B).$$

Proposition 6.5

Two events A and B of non null probability are *independent* *iff*

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B)$$

If A and B are independent, then A^c and B are too also. Similarly for A and B^c, for A^c and B^c.