Probability and Statistics

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Chapter 1

Probability Spaces

1.1 Generalities

Probability calculus aims to give a quantitative analysis of random phenomena, which may appear somehow contradictory, that is proceed to the mathematical analysis of phenomena where randomness plays an important role.

Intuitively, a random phenomena, when repeated a certain number of times with identical conditions, will behave differently, so that the result of that experiment changes from the last one in an imprevisible manner.

For example: game of heads and tails (tossing of coins), dice throwing, lifetime of an electric bulb, arrival time of a sailing boat ...

We can then say that an experiment $\mathcal{E}$ is random if, repeated with identical conditions, can lead to possibly different results, of which we cannot assert for certain the result in advance.

The space or the set of all possible results (outcomes) is usually called:

the state space,

the realizations space,

the space of events,

the samples space.

It is denoted by Ω.

A possible result will be denoted by ω. Thus $\omega \in \Omega$.

This is what is usually called an elementary event.

More generally, an event is a subset of the space of events Ω.

In particuliar, $\emptyset$ is the impossible event and Ω is the certain event.

If Ω is a finite set with $Card = n$, then there are 2^n events.

Example 1.1.1

throwing two coins with head and tail: $\Omega = \{HH, TT, TH, HT\}$. $\odot$

Example 1.1.2

throwing a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$.$\odot$

Example 1.1.3

> *throwing a dart on a circular target of 30cm of diameter and the experiment should describe the impact of the dart in an o.n. basis with center the center of the target:* $\Omega = \{(x,y), x^2+y^2 \leq 15^2\}$. ⊙

Example 1.1.4

> *Lifetime of an electric bulb:* $\Omega = [0, +\infty[$. ⊙

In a random experiment, one and only one elementary event occurs.

Thus, elementary events are non compatible: that is elementary events cannot occur at the same time, and they are also exhaustive, that is Ω is exactly the union of all elementary events.

The realizations space can be finite or countably infinite: we then say that it discrete.

It could be also non countable infinite: we then say that it is continuous.

Example 1.1.5

> *To get familiar with this vocabulary, we may keep in mind the example of a throw of a dice with 6 sides.*
>
> *a) A possible result, for example 5, is a realization. It is denoted by* ω.
>
> *b) All possible events, here* $\{1,2,3,4,5,6\}$, *form the realizations space or the state space. It is denoted by* Ω.
>
> *c) An event is* $A \subset \Omega$. *The opposite event is denoted by* A^c.
>
> *d) We define the certain event as being* Ω;
>
> *e) the impossible event is* $\emptyset$;
>
> *f) the event A and B is* $A \cap B$;
>
> *g) the event A or B is* $A \cup B$;
>
> *h) we say that two events A and B are incompatible if* $A \cap B = \emptyset$.
>
> ⊙

1.2 The concept of a probability

Let us take the example of a throw of a dice with 6 sides.

We want to estimate the probability (that is the chance) to get ”2”. For this purpose, we proceed to a large number N of throws (identical ones) and we count the number of ”2” out.

Denote by:

A: the (elementary) event: getting ”2”;

N(A): number of ”2” out.

We observe that the empirical frequency of success $\frac{N(A)}{N}$ is close to $1/6$ (if the dice is well balanced), and then we can denote it by $P(A)$, that is the ”probability” to obtain A.

The above facts can be generalized to any event A, so that we may define $P(A)$ for all A, by using these empirical frequencies.

In particuliar, we observe that

$$P(\Omega) = 1, \; P(\emptyset) = 0,$$

and for two incompatible events A and B (that is such that $A \cap B = \emptyset$), we have

$$P(A \cup B) = P(A) + P(B).$$

Note that the function P so defined for all events, that it is a function from $\mathcal{P}(\Omega)$ into $\mathbb{R}$.

This is not always the case, that is the initial domain needs not be the full set $\mathcal{P}(\Omega)$, but only a subset $\mathcal{T}$ of it. But we will not give details in this lecture.

In the following, one can keep in mind that $\mathcal{T}$ is $\mathcal{P}(\Omega)$.

Definition 1.2.1

A probability is a function P from $\mathcal{T}$ into $[0,1]$ such that:

- $P(\Omega) = 1$ et $P(\emptyset) = 0$.

- If A_i, $i \in I$, is an at most countable familly of two by two disjoint events, then (σ-additivity)

$$P(\cup_{i \in I} A_i) = \sum_{i \in I} P(A_i)$$

We then say that $(\Omega, \mathcal{T}, P)$ is a probability space.

The event A is said to be almost certain if $P(A) = 1$ and negligable if $P(A) = 0$.

Example 1.2.1

Throwing a dice: $\Omega = \{1, 2, 3, 4, 5, 6\}$, $\mathcal{T} = \mathcal{P}(\Omega)$ and $P(\{\omega\}) = P(\omega) = 1/6$, for all $\omega \in \Omega$. ⊙

Example 1.2.2

$\Omega = \mathbb{R}$ and $\mathcal{T}$ constructed by finite or countable union or intersection of intervals of the type $[a, b]$, with $(a, b) \in \mathbb{Z}^2$. ⊙

Example 1.2.3

$\Omega = \mathbb{N}$ and $\mathcal{T} = \{A, B, \Omega, \emptyset\}$, with $A = \{2n, n \in \mathbb{N}\}$ et $B = \{2n + 1, n \in \mathbb{N}\}$. ⊙

are two examples where $\mathcal{T} \neq \mathcal{P}(\Omega)$.

Proposition 1.2.1

For any events A and B in $\mathcal{T}$, we have:

1. $P(A^c) = 1 - P(A)$;
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
3. $P(A) \leq P(B)$ *if* $A \subset B$;
4. *If A_i, $i \in I$, is an at most countable family of two by two disjoint events, and covering Ω, then*

$$P(B) = \sum_{i \in I} P(A_i \cap B)$$

Proof 1

Exercice.

Example 1.2.4

Ω is the set of points of the sphere (surface of the ball) of radius R in $\mathbb{R}^3$. In that case, $\mathcal{T}$ is made of sufficiently regular subsets of Ω. We can define the probability to find a fly X on a surface element S of that sphere by

$$P(X \in S) = \frac{1}{4\pi R^2}|S|$$

where $|S|$ is the area of the surface element S. This is so if we identify the fly to be a point. Otherwise, we need to take into account the contact surface between the fly and the sphere and to take into account the scales. $\odot$

1.3 Case finite ou countably infinite

If Ω is at most countable, then we take $\mathcal{T} = \mathcal{P}(\Omega)$. It is then clear that the probability P is completely determined by the values $P(\omega)$, for all $\omega \in \Omega$.

Indeed, one can show that, for all events $A \in \mathcal{T}$, we have

$$P(A) = \sum_{\omega \in A} P(\omega)$$

Example 1.3.1

We take $\Omega = \{\omega_1, \omega_2\}$, thus containing two elements, with $P(\omega_1) = p$ and $P(\omega_2) = 1 - p = q$. For example, this is the model used for the experiment "throw of coin (head or tail)", or more generally for a random experiment with two possible outcomes only (ny first kid will be a boy or a girl?).

In case Ω is finite, the most important example of a probability on Ω is the uniform probability:

Definition 1.3.1

If Ω if finite, we call uniform probability on Ω the probability P defined by

$$P(\omega) = \frac{1}{card\ \Omega}, \text{ for all } \omega \in \Omega$$

In that case, for any event A, we have

$$P(A) = \frac{card\ A}{card\ \Omega}$$

1.3.1 Counting

We recall:

- the number of permutations (or bijections) of $\{1, ..., n\}$ is $n!$.

- the number of arrangements of k elements from n, or the number of injections from $\{1, ..., k\}$ into $\{1, ..., n\}$ is $A_n^k = \frac{n!}{(n-k)!}$.

- the number of subsets with k elements in a set with n elements is $C_n^k = \frac{n!}{k!(n-k)!}$.

- We have the binomial formulae:

$$(x+y)^n = \sum_{k=0}^{n} C_n^k x^k y^{n-k}$$

Example 1.3.2

Let us consider a group of n students. We assume there is no leap years. We want to compute the probability p_n to have that two students at least have the same anniversay day.

For that purpose, we define the probability space firstly: $\Omega = \{1, ..., 365\}^n$; here $\omega = (\omega_1, ..., \omega_n)$, where ω_i is the anniversary day of the student i. We choose the uniform probability on Ω (which is far from being a good choice). We denote by A the event "at least two students have the same anniversary day". Thus we have

$$p_n = P(A) = 1 - P(A^c)$$

On the other hand, A^c is the event "all students have different anniversary day". That is $A^c = \{\omega, \omega_i \neq \omega_j, \forall i \neq j\}$. Its number of elements is the number of injections from $\{1, ..., n\}$ into $\{1, ..., 365\}$. We find finally that

$$p_n = 1 - \frac{365!}{(365-n)!365^n} \text{ if } n \leq 365 \text{ and } 1 \text{ otherwise.}$$

For example, we find that

$$p_{22} \simeq 0,476, \; p_{23} \simeq 0,507, \; p_{366} = 1.$$

$\odot$

1.3.2 Drawing and Urns

This section is difficult. We suggest, while doing an exercice, to compute from scratch the results. Do not try to remember the results of this section. However, keep in mind the different types of drawing.

Let N balls with k different colors: N_1 balls with color 1, ..., N_k balls with color k.

We define the proportion of balls of color i by

$$p_i = N_i/N.$$

Experiment: we draw at random n balls from the urn, with $n \leq N$.

Problem: we want to consider the distribution (frequency) of obtained colors, and more precisely the number

$$P_{n_1 n_2 \ldots n_k}$$

which is the probability to obtain n_1 balls of color 1..., with $n_1 + n_2 + ... + n_k = n$.

Here we need to precise the concept of drawing: drawing with reset, drawing without reset, simultaneous drawing.

a) Simultaneous drawing

We draw all balls in the same time.

Here, Ω is the set of all sets of length n of distinct elements, among N, thus whose number is C_N^n.

Since the number of cases giving the requested distribution of colors is

$$C_{N_1}^{n_1}...C_{N_k}^{n_k},$$

we deduce that

$$P_{n_1 n_2 ... n_k} = \frac{C_{N_1}^{n_1}...C_{N_k}^{n_k}}{C_N^n}$$

This is called the polygeometric distribution.

When we have exactly two colors,

$$P_{n_1, n-n_1} = \frac{C_{N_1}^{n_1} C_{N-N_1}^{n-n_1}}{C_N^n}$$

is the hypergeometric law.

Example 1.3.3

> *In a production in series, we know that among N machined components, M are defective. If we take at random a sample of n components, then the probability that this sample contains k defective components is*
>
> $$\frac{C_M^k C_{N-M}^{n-k}}{C_N^n}.$$
>
> ⊙

b) Drawing with reset

Here the drawings are successively done, and with reset of the drawn ball each time.

Ω is thus the set of n-couples of elements from the urn. Thus *Card* $\Omega = N^n$, and we take the uniform probability over Ω.

The number of n-couples with distribution n_1, ...,n_k is:

$$\frac{n!}{n_1!n!...n_k!} N_1^{n_1}...N_k^{n_k}.$$

Indeed, the number of ways to fix the locations of the k colors among n is equal to the number of ways to divide n into k parts of size n_i, thus explaining the first factor. Then, once the location of colors is fixed, we have N_i possibilities for each ball of color i. Thus

$$P_{n_1...n_k} = \frac{n!}{n_1!n!...n_k!} \frac{N_1^{n_1}...N_k^{n_k}}{N^n}.$$

This is the multinomial distribution.

If $k = 2$, $p_1 = p$ and $p_2 = 1 - p$, we obtain the probability

$$P_{n_1, n-n_1} = C_n^{n_1} p^{n_1} (1-p)^{n-n_1}.$$

which is the binomial law with parameters n and p.

c) Drawing without reset

We draw the balls successively, but without reset. Ω is the set of all sequences of n distinct elements among N, whose number is A_N^n.

One can show that we obtain the same probability as in the simultaneous drawing.

Thus we have equivalence between drawing without reset and simulatneaous drawing.

Example 1.3.4

We draw at random 4 cards from a deck of 52 cards. We want to know the probability that, among these 4 cards, there is exactly 2 kings.
For that purpose, we take Ω the set of parts with 4 elements of 52 cards.... we are in the case "simultaneous drawing" .... ⊙

Example 1.3.5

Let 20 components of type I, among which 5 are defective, and 30 components of type II, among which 15 are defective. We want to build a system composed of 10 components of type I and of 5 components of type II, placed in series. We want to compute the probability for the system to operate, the components being choosen at random.

Here, the number of different systems that we may build is $C_{10}^{20} \times C_5^{30}$. The number of different systems which could operate is $C_{10}^{15} \times C_5^{15}$. Thus, using equiprobability, the sought probability is

$$\frac{C_{10}^{15} \times C_5^{15}}{C_{10}^{20} \times C_5^{30}} \simeq 0,00034.$$

⊙

Example 1.3.6

[Difficult; to be read if you have time] *Let be given n particles and $m > n$ boxes (which could be thought as energy levels). We put at random each particle in a box. We want to find the probability p that in n selected boxes, one and only one particle could be found.*

We consider three types of realizations.

1. Maxwell-Boltzmann Statistics. If we accept as possible outcomes all the ways to put n particles in n boxes, and distinguishing each particle, then

$$p = \frac{n!}{m^n}.$$

2. Bose-Einstein Statistics. If we assume that particles could not be distinguished, then

$$p = \frac{(m-1)!n!}{(n+m-1)!}.$$

3. Fermi-Dirac Statistics. If we do not distinguish particles and if we assume that in each box, we may put at most one particle, then

$$p = \frac{n!(m-n)!}{m!}.$$

⊙

1.4 Geometric Probabilities

Let Ω be a regular and bounded subset of $\mathbb{R}^n$. One often uses the following probability

$$P(A) = \frac{|A|}{|\Omega|}$$

where $|A|$ and $|\Omega|$ are the measure of these subsets, that is the length, area or volume in dimension 1, 2 ou 3 respectively. Then, we construct $\mathcal{T}$ by finite intersection and union, at most in a countable way, of blocks.

Example 1.4.1

$\Omega = [a, b] \subset \mathbb{R}$*; we take*

$$P([c, d]) = \frac{d-c}{b-a}.$$

⊙

Example 1.4.2

Two persons select at random one point in $[0, 1]$*. Let* x *and* y *be the outcomes of these experiments, We want to compute the probability to have* $|x - y|$ *bigger than* u*, where* u *is a fixed value in* $[0, 1]$*.*

We take the following model: $\Omega = [0, 1] \times [0, 1]$*,* $P(A) = |A|$*,* A *being a subset of* Ω *of area* $|A|$*. A point of* Ω *(experiment outcome) satisfies* $|x - y| > u$ *if it belongs to* D_1 *or to* D_2*, where* D_1 *and* D_2 *are the corners of the unit square (draw a picture) of diagonal length* u*. The event* A *whose probability is looked after, is then identified with* $D_1 \cup D_2$ *of area* $(1 - u)^2$*. Thus*

$$P(A) = (1-u)^2.$$

⊙

Example 1.4.3

Compute the probability P *for a point choosen at random, inside a sphere of radius* R*, to be closer to the center than to the surface of the sphere?*

We find

$$P = \frac{\frac{4}{3}\pi(\frac{R}{2})^3}{\frac{4}{3}\pi R^3} = \frac{1}{8}.$$

⊙

Example 1.4.4

Bertrand paradox. Let be given a circle of radius r. We want to compute the probability p to have the length l of a segment AB, with A and B taken on the circle, the segment being picked at random, bigger than the length $r\sqrt{3}$ of the inner equilateral triangle.

In fact, we have at least three solutions.

1. If the center M of the cord AB is inside the circle C_1 of radius $r/2$ (with the same center as C), then $r > r\sqrt{3}$. We can then expect cases as being favorable as those cases where these points are inside C_1, and as all possible outcomes, the set of points inside C_1. Using geometric probabilities, we can deduce that

$$p = \frac{\pi r^2/4}{\pi r^2} = \frac{1}{4}.$$

2. We assume that the endpoint A is fixed. This reduces the number of outcomes but it has no effect on the value of p because the number of possible positions for B is then consequently reduced. If B is on the cord of $120°$, then this is ok, that is the farovable outcomes. We find

$$p = \frac{2\pi r/3}{2\pi r}.$$

3. Lastly, we assume that the direction AB is orthogonal to a fixed diameter FK. If the center M of AB is between G and H, this is ok, that is the favorable cases. Then we get

$$p = \frac{r}{2r} = \frac{1}{2}.$$

⊙

1.5 Conditional Probabilities

Let us take the example of throwing two well balanced dices.

We want to compute the probability of:

A: "the sum of the two dices is bigger than 10",

knowing that

B: "the second gives 5".

We find that the empirical frequency is

$$\frac{N(A \cap B)}{N(B)} = \frac{N(A \cap B)}{N} \frac{N}{N(B)},$$

that is $\frac{P(A \cap B)}{P(B)}$.

We deduce that an additional à priori information changes the sought probability.

More generally, if Ω is the set of events associated to a random experiment $\mathcal{E}$, if A and B are two events, and if we assume that when performing this experiment, that B has occured, then B becomes the new space of events, and for A to occur, we must have that $A \cap B$ has occured.

Definition 1.5.1

Let A and B be two events, with $P(B) \neq 0$. We call ***probability (conditional)*** *of A knowing B, the number*

$$P(A|B) \equiv \frac{P(A \cap B)}{P(B)}.$$

If we set $P_{|B}(A) = p(A|B)$, then one can show that $P_{|B}$ satisfies the axioms of a probability. For example, we can write

$$P(A^c|B) = 1 - P(A|B).$$

One can also show that

Proposition 1.5.1

Multiplication Rule *If $P(B) \neq 0$, then*

$$P(A \cap B) = P(A|B)P(B)$$

and if $P(A) \neq 0$, then

$$P(A \cap B) = P(B|A)P(A)$$

Example 1.5.1

Two components picked at random (one by one) and without reset from a box containing ten with brand A and ten with brand B. What is the probability to get a) two components with brand A? b) two components with same brand? c) two components with different brand?

Let A_k: a component with brand A is obtained at drawing number i. We look for

$$P(A_1 \cap A_2) = P(A_2|A_1)P(A_1) = \frac{9}{19} \times \frac{10}{20} = \frac{9}{38}.$$

We may also use the multiplication rule to write

$$P(A_1 \cap A_2) = P(A_1|A_2)P(A_2).$$

but we cannot use this formula to reach the result.

b) the problem is symmetric, because there are as many components of brand A than brand B, the preceding result gives that the sought probability is $2.\frac{9}{38} = \frac{9}{19}$.

c) we deduce from b) that the sought probability is $1 - \frac{9}{19} = \frac{10}{19}$.

$\odot$

Proposition 1.5.2

Bayes formula *If $P(A).P(B) \neq 0$, then*

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Definition 1.5.2

Partition *Let B_1, B_2, ..., B_n be a sequence (eventually countable) of two by two disjoint events and whose union covers the events space Ω. This is called a partition of Ω.*

We often assume that these events are of non zero probabilities. The most simple example is given by A and A^c. One can show

Proposition 1.5.3

Total Probability Rule *If $(B_i)_{i \in \mathbb{N}}$ is a partition of Ω, then for all $A \subset \Omega$, we have*

$$P(A) = \sum_{k \in \mathbb{N}} P(A \cap B_k) = \sum_{k \in \mathbb{N}} P(A|B_k)P(B_k)$$

if $P(B_k) > 0$ for all k.

Finally, we get

Proposition 1.5.4

Bayes Rule

Let $(B_i)_i$ be a partition of Ω with nonzero probabilities. Then for all $A \subset \Omega$, we have

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_k P(A|B_k)P(B_k)} \text{ for all } j$$

Example 1.5.2

Let consider a communication system which emits either a 0, or a 1. Because of noise, the emitted signal is incorrectly received. We define the events

$$E_i : i \text{ is emitted and } R_i : i \text{ is received}$$

for $i = 0$ and 1. We assume that $P(R_0|E_0) = 0{,}7$, $P(R_1|E_1) = 0{,}8$ and that the 0 is emitted 60% of time.

a) compute $P(E_0|R_1)$

b) compute the probability to have a transmission error.

a) We have

$$P(E_0|R_1) = \frac{P(R_1|E_0)P(E_0)}{P(R_1|E_0)P(E_0) + P(R_1|E_1)P(E_1)}$$

$$= \frac{(1-0{,}7)(0{,}6)}{(1-0{,}7)(0{,}6) + (0{,}8)(0{,}4)} = 0{,}36.$$

b)

$$P(\text{transmission error }) =$$

$$= P(E_0 \cap R_1) + P(E_1 \cap R_0) = P(R_1|E_0)P(E_0) + P(R_0|E_1)P(E_1)$$

$$= (1-0,7)(0,6) + (1-0,8)(0,4) = 0,26.$$

Note that the events E_0 and E_1 form a partition of Ω. This is so also for R_0 and R_1. $\odot$

Definition 1.5.3

Two events A and B are called independent iff

$$P(A \cap B) = P(A)P(B).$$

Remark 1.5.1

Two independent events may or not be incompatible. If they are independent and incompatible, then $P(A)$ or $P(B)$ or both are zero.

Proposition 1.5.5

Two events A and B wth non zero probability are independent iff

$$P(A|B) = P(A) \text{ or } P(B|A) = P(B)$$

If A and B are independent, then A^c and B are too. Similarly for A et B^c, for A^c et B^c.

Example 1.5.3

In a factory, 96% of manufactured computers comply to official standards. Each computer is assessed to two independent control schedules. We assume that each of these operations assess good 98% of units which are effectively good, and 6% of units which are not compliant to the standards. Compute the probability that a delivered unit is effectively good.

Let
A: the unit satisfied the control schedules.
B: the unit is good.
We look for

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

$$=_{ind} \frac{(0,98)^2(0,96)}{(0,98)^2(0,96) + (0,06)^2(0,04)} \simeq 0,9998.$$

Note that $B_1 = B$ and $B_2 = B^c$ form a partition of Ω. If A_k denotes : the unit has satisfied the control schedule number k, then we can write $A = A_1 \cap A_2$. Note that A_1 and A_2 are conditionally independent wrt B and wrt B^c but are not independent. $\odot$

Example 1.5.4

A person competes in a game show. At the end, he is in front of three doors and he has to choose one. Behind one of them, is hidden the big jackpot, put there at random. There is nothing behind the other two. The game's host knows where is hidden the jackpot. Assume that the person *chooses door 1 and that the host tells him that the person was right not to choose door number 3 because there is nothing behind that one. He then offers him the possibility to change the door, and so to choose door number 2. What is the possibility that the person wins the jackpot, if she decides to keep door 1?*

Let A_k: the jackpot is behind door k for $k = 1, 2, 3$. Let F: the host eliminates door 3. Assume that if the person chooses the good door, then the host eliminates door 3 with probability $1/2$. In that case,

$$P(F) = P(F|A_1)P(A_1) + P(F|A_2)P(A_2) + P(F|A_3)P(A_3) = \frac{1}{2}\frac{1}{3} + 1\frac{1}{3} + 0 = \frac{1}{2}$$

thus

$$P(A_1|F) = \frac{P(F|A_1)P(A_1)}{P(F)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Thus the person has a probability of $2/3$ to win the jackpot if she decides to change the door. In general, if there are n doors, and if the host eliminates $n-2$ (he does not eliminate the one choosen by the person), then the probability that the jackpot is hidden behind the only remaining door, among the $n-1$ doors still in play, is $(n-1)/n$. ⊙

Example 1.5.5

(The liars) A says that B told him that C has lied. If the three persons are saying the truth with a probability $p \in (0, 1)$, and so independently of each person, what is the probability that C has been effectively lying?

Let F: A says that B told him that C has lied, and F_I: I lied, for $I = A, B, C$.

We look for

$$P(F_C|F) = \frac{P(F|F_C)P(F_C)}{P(F|F_C)P(F_C) + P(F|F_C^c)P(F_C^c)}.$$

Or on a

$$P(F|F_C) = P(F_A^c \cap F_B^c) + P(F_A \cap F_B) =_{ind} p^2 + (1-p)^2$$

et

$$P(F|F_C^c) = P(F_A^c \cap F_B) + P(F_A \cap F_B^c) =_{ind} 2p(1-p)$$

Donc

$$P(F_C|F) = \frac{[p^2(1-p)^2](1-p)}{[p^2 + (1-p)^2](1-p) + [2p(1-p)]p} = \frac{p^2 + (1-p)^2}{3p^2 + (1-p)^2}.$$

Note that if $p = 1/2$, then we find $1/2$ which is quite reasonable. ⊙

1.6 Exercices

1. Let P be a probabily defined over a finite set with 4 elements $\Omega = \{a, b, c, d\}$. We choose $\mathcal{T}$ as being the set of all subsets of Ω. Compute $P(a)$ for each of the following cases:

   (a) $P(b) = 1/4, P(c) = 1/6, P(d) = 1/5;$

   (b) $P(a) = 3P(b), P(c) = P(d) = 1/4;$

   (c) $P(\{b, c, d\}) = 2p(a);$

   (d) $P(b) = P(a), P(c) = 2p(b), P(d) = 3P(c).$

   <u>Hints</u>: Use the additivity of a probability. (a) $P(a_1) = \frac{23}{60}$. (b) $P(a_1) = \frac{3}{8}$. (c) $P(a_1) = \frac{1}{3}$. (d) $P(a_1) = \frac{1}{10}$.

2. Consider events A and B such that

   $$P(A) = 1/2, P(A \cup B) = 3/4 \text{ et } P(\bar{B}) = 5/8$$

   Find $P(A \cap B)$, $P(\bar{A} \cap \bar{B})$, $P(\bar{A} \cup \bar{B})$ et $P(B \cap \bar{A})$.

   <u>Hints</u>: Use probability of an union, complementary set, Morgan laws. $P(A \cap B) = \frac{1}{8}$, $P(\bar{A} \cap \bar{B}) = \frac{1}{4}$, $P(\cap A \cup \bar{B}) = 0,875$, $P(B \cap \bar{A}) = \frac{1}{4}$.

3. In a lottery, 5 balls are drawed, at random and without reset, among 25 balls numbered from 1 to 25. We get the jackpot if the 5 balls are picked with the indicated order.

   (a) What is the probability to gain the jackpot?

   (b) What is the probability not to gain the jackpot because of only one ball?

   <u>Hints:</u> a) $P(jackpot) = \frac{1}{A_5^{25}} = \frac{20!}{25!} = ... = \frac{1}{6.375.600}$; b) the number of outcomes where only one ball is wrong is given by $C_1^5 \times 20$. Thus the sought probability is $\frac{100}{6.375.600}$.

4. (Car) Licence plates are made with three letters followed by four numbers (from 0 to 9). We asume that letters I and O are never used and that no plate is made using the number 0000.

   (a) How many different plates could we get?

   (b) What is the answer in a), if morever, no plate contains three identical letters, nor four identical numbers?

   <u>Hints:</u> a) the number of identical plates is

   $$(24 \times 24 \times 24) \times (10^4 - 1) = 138.226.176$$

   b) in that case, the total number of different plates is given by

   $$(24^3 - 24) \times (10^4 - 10) = (13.800)(9990) = 137.862.000$$

5. We throw p well balanced dices with n sides numbered from 1 to n. What is the probability to get:

   (a) exactly one 1?
   (b) at least one 1?
   (c) at most one 1?
   (d) exactly two 1?

   Hints: Here Ω is the set of ordered p-uplets of numbers between 1 and n (eventually with repetition). Thus $card\Omega = n^p$. (a) the number of favorable cases is $C_p^1(n-1)^{p-1}$, as there are C_p^1 ways to choose the location of 1 among the p numbers of the p-uplet, then, once this location choosen, the other locations should be numbers between 2 and n, whose number is $n-1$. (b) Using complementary event, the number of p-uplets without 1 is $(n-1)^p$, thus the number of favorable cases is $n^p - (n-1)^p$. (c) we count the number of p-uplets with only one 1, that is $p(n-1)^{p-1}$ and the number of p-uplets without 1, that $(n-1)^p$, and we add. (d) The number of favorable cases is $C_p^2(n-1)^{p-2}$. We may also use the model of urn with two categories (here with or without 1, with reset).

6. Compute the probability for the sum of numbers choosen at random in $[0,1]$ not to exceed 1 and the product be less than 2/9.

   Indications: Here Ω is the set of couples (x,y) in $[0,1]$, with the geometric probability, that is the area. We want $x+y \leq 1$ and $xy \leq \frac{2}{9}$. The area is $\frac{1}{3} + \frac{2}{9}\log 2$.

7. We throw two well balanced dices with 6 sides. Compute the probability for:

   (a) the sum of obtained numbers on the two dices to be bigger than 9, knowing that we have at least one 6.
   (b) to have one 4 on a dice, knowing that we have at least one 2.
   (c) the sum of obtained numbers of the two dices to be 5 knowing that the difference between the biggest and lowest values is equal to 4.

   Hints: Here Ω is the set of ordered sequences of two numbers taken with repetition among $\{1, ..., 6\}$. Denote by S the sum and by D the difference. (a) Let $A =$ "at least one 6". Then

$$P(S \geq 9|A) = \frac{P(S \geq 9 \cap A)}{P(A)}$$

   and we find by counting $\frac{7/36}{11/36}$. (b) Similarly, we find $\frac{2/36}{11/36}$. (c) We have $card\ "D = 4" = 4$. And $card\ "S = 5" \cap "D = 4" = 0$. Therefore the probability equals 0.

8. A class of students has 10 boys, half of which have brown eyes, and 20 girls, half of which with brown eyes too. Compute the probability that a student taken at random:

   (a) is a boy;
   (b) has brown eyes;
   (c) is a boy or has brown eyes.

   Hints: No serious issues.

9. Let A and B events such that $P(A \cap B) = P(A^c \cap B) = P(A^c \cap B) = P(A \cap B^c) = p$. Compute $p(A^c \cap B^c)$ and $P(A^c \cup B^c)$.

   Hints: $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - 3p$ and $P(A^c \cup B^c) = 1 - P(A \cap B) = 1 - p$.

10. A communication system transmits 3 signals, s_1, s_2 et s_3, with the same probability. Received signals could be inaccurate, because of noise. We find, experimentally, that the probability p_{ij} for receiving signal s_j, knowing that signal s_i was emitted is given by the following table: on row (line) emitting, and in column, receiving

	s_1	s_2	s_3
s_1	$0,8$	$0,1$	$0,1$
s_2	$0,05$	$0,90$	$0,05$
s_3	$0,02$	$0,08$	$0,90$

   (a) What is the probability that signal s_1 was emitted, knowing that signal s_2 has been received?

   (b) If we assume that emitting signals are independent, what is the probability to receive two signals s_3 consecutively?

   Hints: Let E_i: signal s_i was emitted, and R_i: signal s_i has been received, for $i = 1, 2, 3$. a) we look for

$$P(E_1|R_2) = \frac{P(R_2|E_1)P(E_1)}{P(R_2)} = \frac{(0,1)(1/3)}{0,36} \simeq 0,0926$$

   as

$$P(R_2) = \sum_{i=1}^{3} P(R_2|E_i)P(E_i) = \frac{1}{3}[0,1 + 0,9 + 0,08] = 0,36$$

   b) we have

$$P(R_3) = \sum_{i=1}^{3} P(R_3|E_i)P(E_i) = \frac{1}{3}[0,1 + 0,05 + 0,90] = 0,35$$

   Then, using independence, the probability to receive two consectuive signals s_3 is given by $(0,35)^2 = 0,1225$.

11. From collected data, it appears that 40% of human beings have type A blood, 10% type B, 45% type O and 5% type AB. Moreover, we know that 90% of persons of type O will be checked correctly, while 3% of persons of type B, 10% of type AB and 2% of type A will be checked as type O.

   (a) What is the probability that a person of type O will be effectively of this type?

   (b) If we assume independence of events, what is the probability that two given persons of type O are not of this type?

   Hints: Let O: the person is of type O, C_0: the person is checked as being of type O, similarly for the other types.

   a) We look for

$$P(O|C_O) = \frac{P(C_O|O)P(O)}{P(0)} = \frac{(0,90)(0,45)}{0,4210} \simeq 0,9620$$

as

$$P(C_O) = P(C_O|A)P(A) + ... + P(C_O|AB)P(AB) = 0,4210$$

b) we have $P(O^c|C_O) = 1 - P(O|C_O) \simeq_{a)} 0,0380$. Then, using independence, the sought probability is $(0,0380)^2 \simeq 0,0014$.