Chapter 1- Line Integrals

1 Basic facts on paths

Definition 1.1 *A path of* $I\!R^n$ *is a map* $c : I \to I\!R^n$, *with* $I = [a, b]$. *The subset of* $I\!R^n$ $\mathcal{C} = c([a, b])$ *is called the curve parametrized by the path* c. $c(a)$ *and* $c(b)$ *are the enpoints of the curve* $\mathcal{C}$. *We way also that* c *is a parametrization of the curve* $\mathcal{C}$.

Example 1.1 *Let* $c : [0, 1] \to I\!R$ *be given by*

$$c(t) = (x_0, y_0, z_0) + tv$$

where (x_0, y_0, z_0) *is a fixed point of* $I\!R^3$ *and* v *a non null vector of* $I\!R^3$. *Then the curve associated with this path* c *is the segment of* $I\!R^3$ $[(x_0, y_0, z_0); (x_0, y_0, z_0) + v]$.

Example 1.2 *Let* $c : [0, 2\pi] \to I\!R^2$ *be given by*

$$c(t) = (\cos t, \sin t)$$

Then the associated curve $\mathcal{C}$ *is the unit circle of* $I\!R^2$.

Definition 1.2 *If* c *is a continuous map, derivable (or differentiable) ..., we say that the path* c *is continuous, derivable, ... The velocity vector at time* t *at point* $c(t)$ *is the vector* $c'(t)$. *The speed at time* t *and point* $c(t)$ *is the norm of this vector, that is* $\| c'(t) \|$.

Remark 1.1 *The velocity* $c'(t)$ *is a vector tangent to the path* c. *We say also that it is tangent to the curve* $\mathcal{C}$ *parametrized by* c.

2 Path integrals

Definition 2.1 *Let* $c : [a, b] \to I\!R^n$ *be a path of class* C^1 *(by pieces). We call length of* c *the number*

$$L(c) \equiv \int_a^b \| c'(t) \| \, dt$$

More generally, we have the following definition

Definition 2.2 *With the same previous assumptions, let moreover* $f : I\!R^n \to I\!R^n$ *be a continuous function (at least in a neighborhood of* $\mathcal{C}$*). Then we define*

$$\int_c f(s)ds \equiv \int_a^b f(c(t)) \| c'(t) \| \, dt$$

Cleraly if $f = 1$, we recover the length of the path c.
Let us try to explain why the length as defined above coincides with the usual one (taking into account the speed). To simplify, let us work with a path $c : [a, b] \to \mathbb{R}^3$. The idea is that to compute the length of c, we are going to make polygonal approximations. Let us consider a subdivision of order N of $[a, b]$. Consider the polygonal lines based on points $c(t_i)$, where

$$a = t_0 < t_1 < \ldots < t_N = b,\ t_{i+1} - t_i = \frac{b-a}{N}, 0 \leq i \leq N-1$$

Then the idea is to say that the length of c will be almost equal to the length of this broken line, for N large enough, that is

$$S_N = \sum_{i=0}^{N-1} \parallel c(T_{i+1}) - c(t_i) \parallel$$

If $c(t) = (x(t), y(t), z(t))$, applying the intermediate value theorem from calculus, for $x(t)$, $y(t)$ and $z(t)$ on the interval $[t_i; t_{i+1}]$, there exists t_i^*, t_i^{**}, t_i^{***} such that

$$x(t_{i+1}) - x(t_i) = x'(t_i^*)(t_{i+1} - t_i)$$

$$y(t_{i+1}) - y(t_i) = y'(t_i^{**})(t_{i+1} - t_i)$$

$$z(t_{i+1}) - z(t_i) = z'(t_i^{***})(t_{i+1} - t_i)$$

And thus, we obtain

$$S_N = \sum_{i=0}^{N-1} \sqrt{x'(t_i^*)^2 + y'(t_i^{**})^2 + z'(t_i^{***})^2}(t_{i+1} - t_i)$$

Formally when N goes to $+\infty$, the polygonal line will be closer and closer to the curve $\mathcal{C}$ and thus we recognized in the expression of S_N the Riemann sum associated wth the integral which defines the length of c.
The motivation for the definition of path integrals is done similarly.
Here is an important case. Assume that $\mathcal{C}$ is a plane curve, for example contained in the plane $(x0y)$ of $\mathbb{R}^3$. Let f be a positive function of variables x and y. Then $\int_c f(s)ds$ is the area of the lateral surface.

3 Line integrals

Let $\vec{F}$ be a forces field of $\mathbb{R}^3$ that is a map from $\mathbb{R}^3$ into $\mathbb{R}^3$. Assume that this field acts on a given particle of $\mathbb{R}^3$. Moreover assume more precisely that this particle is moving along a fixed curve $\mathcal{C}$ and that is submitted to this force field $\vec{F}$. We want to compute the work done by this fielf $\vec{F}$ on this particle. In fact we want also to give a definition of what is the work. If $\mathcal{C}$ is a line segment, for example, directed by a vector $\vec{d}$, with $\| \vec{d} \|$ being the length of $\mathcal{C}$, then the corresponding work is given by

$$T_{\mathcal{C}}(\vec{F}) = \vec{F}.\vec{d}$$

This fits the intuitive notion: for a particle moving on this segment with the uniform velocity $\vec{d}$, the work will be bigger when the force is directed in the same direction as $\vec{d}$. Note that it is null when $\vec{d}$ and $\vec{F}$ are orthogonal.

If now, $\mathcal{C}$ is no more a line segment, we'll define the work by the formula

$$T_{\mathcal{C}}(F) = \int_a^b \vec{F}(c(t)).c'(t)dt$$

where c is a parametrization of the curve $\mathcal{C}$ and giving the position of the particle on the curve $\mathcal{C}$.

Let us explain why this formula is (almost) correct. Firstly, note that in the case where $\mathcal{C}$ is a line segment, we recover the first formula. In the general case, assume $t \in [t, t + \Delta t]$ with Δt small. Then the particle will move from $c(t)$ to $c(t + \Delta t)$. The corresponding displacement vector is thus

$$\vec{\Delta s} = c(t + \Delta t) - c(t) \simeq c'(t)\Delta t$$

The work between $c(t)$ and $c(t + \Delta t)$ will then be almost equal to

$$\vec{F}(c(t)).\vec{\Delta s} \simeq \vec{F}(c(t)).c'(t)\Delta t$$

Now if we divide the interval $[a, b]$ along a regular subdivision of order N as above, with $\Delta t = t_{i+1} - t_i$, the total work done by $\vec{F}$ will be almost equal to

$$\sum_{i=0}^{N-1} \vec{F}(c(t_i)).\vec{\Delta s} \simeq \sum_{i=0}^{N-1} \vec{F}(c(t_i)).c'(t_i)\Delta t$$

and this is a Riemann sum associated with the integral in the above definition.

In conclusion, we introduce the following definition

Definition 3.1 *Let $\vec{F} : I\!R^n \to I\!R^n$ be a force field, continuous in a neighborhood of the path c, $c : [a, b] \to I\!R^n$ (class C^1). Then we define the line integral of $\vec{F}$ along c, or in other words, the work of $\vec{F}$ on c as the number defined by*

$$\int_c \vec{F}.ds \equiv \int_a^b \vec{F}(c(t)).c'(t)dt$$

Example 3.1 $c(t) = (\sin t; \cos t, 0)$, $0 \leq t \leq 2\pi$, *et* $F = (x, y, z)$.

Definition 3.2 *Notation: With the above assumptions and notations, we also denote, if* $\vec{F} = (F_1, F_2, F_3)$, *and* $n = 3$

$$\int_c \vec{F}.ds = \int_c F_1 dx + F_2 dy + F_3 dz$$

Thus if $c(t) = (x(t), y(t), z(t))$, we have

$$\int_c F_1 dx + F_2 dy + F_3 dz = \int_a^b [F_1(c(t))x'(t) + F_2(c(t))y'(t) + F_3(c(t))z'(t)]dt$$

Example 3.2 *Compute* $\int_c x^2 dx + xydy + dz$ *with* $c(t) = (t, t^2, 1)$ *and* $0 \leq t \leq 1$.

Example 3.3 *Compute* $\int_c \cos z dx + e^x dy + e^y dz$ *with* $c(t) = (1, t, e^t)$ *and* $0 \leq t \leq 2$.

It is important to note that $\int_c \vec{F}.ds$ depends on $\vec{F}$ but also on the path c. What is the link with the associated curve $\mathcal{C}$? In particuliar, if we change the parametrization of the same curve, do we change the value of the line integral? More precisely, if c_1 and c_2 are two different paths, but parametrizing the same curve $\mathcal{C}$, do we have

$$\int_{c_1} \vec{F}.ds = \int_{c_2} \vec{F}.ds \ ?$$

It is quite clear that in general the answer is on the negative: indeed, recall that it is linked with the notion of work. A parametrization gives a way to move on the curve. Thus the work depens on the way we move on that curve too.

However, there are cases where

$$\int_{c_1} \vec{F}.ds = \mp \int_{c_2} \vec{F}.ds$$

Definition 3.3 *Let* $h : I \to I_1$ *be a* C^1 *bijective map with* $I = [a, b]$ *and* $I_1 = [a_1, b_1]$. *Let* $c : I_1 \to I\!R^n$ *be a path of class* C^1 *(by pieces). Then* $p = c \circ h : I \to I\!R^n$ *is a parametrization of* c.

In fact we have $Im\ c = \ Im\ p$. Thus c and p are two parametrizations of the same curve $\mathcal{C}$. As we have $p'(t) = c'(h(t)).h'(t)$, we note that :

1) if h is stricly increasing, $h(a) = a_1$ and $h(b) = b_1$.

2) if h is strictly decreasing, $h(a) = b_1$ and $h(b) = a_1$.

Thus, we have one of the two following cases:

1) $c \circ h(a) = c(a_1)$ *and* $c \circ h(b) = c(b_1)$

2) or $c \circ h(a) = c(b_1)$ *and* $c \circ h(b) = c(a_1)$

In the first case, we say that the re-parametrization preserves the orientation, while in the second case, it reverses the orientation.

If we preserve the orientation, a particle moving on $\mathcal{C}$ according to $c \circ h$ will move in the same direction as along c.

Example 3.4 *If* $c : [a, b] \to I\!R^3$ *is* C^1 *then*

$$c_{op} : [a, b] \to I\!R^3, c_{op}(t) = c(a + b - t)$$

is a reparametrization of c *reversing the orientation.*

Theorem 3.1 *Let* $F : I\!R^n \to I\!R^n$ *be a vector field, continuous in a neighborhood of the path of class* C^1 $c : [a, b] \to I\!R^n$*. Let* $p : [a_1, b_1] \to I\!R^n$ *be a reparametrization of* c*. Then*

$$\int_p F.ds = \mp \int_c F.ds$$

with the sign $+$ *if* p *preserves the orientation of* c *and the sign* $-$ *if* p *reverses the orientation.*

Example 3.5 *Let* $f(x, y, z) = (yz, xz, xy)$ *and*

$$c : [-5; 10] \to I\!R^3, \; c(t) = (t, t^2, t^3)$$

We find

$$\int_c F.ds = 984,375 \text{ and } \int_{c_{op}} F.ds = -984,375$$

Theorem 3.2 *(change of parametrization for the path integrals). Let* c *be a path of class* C^1 *(by pieces),* p *any reparametrization of* c *and* $f : I\!R^3 \to I\!R$ *a continuous map. Then*

$$\int_c f ds = \int_p f ds$$

In the case when the field is a gradient, we have

Theorem 3.3 *Assume that* $\vec{F} = \nabla f$*, with* $f : I\!R^3 \to I\!R$ C^1*. Then*

$$\int_c F.ds = f(c(b)) - f(c(a))$$

Example 3.6 *Let* $c(t) = (4^4/4, \sin^3(t\pi/2), 0)$*,* $t \in [0, 1]$*. We want to compute* $\int_c ydx + xdy$*. Here* $F = (y, x, 0)$ *and thus* $F = \nabla f$ *avec* $f = xy$*. Thus ....*

Definition 3.4 *We define a simple curve* $\mathcal{C}$ *as being the image of a path of class* C^1 *(by pieces)* $c : I \to I\!R^n$ *injective on the interior of* I*. We then say that* c *is an adapted parametrization to* $\mathcal{C}$

A simple curve corresponds to a curve which does not self intersect except eventually at the endpoints.
If $I = [a, b]$, $c(a) = P$ and $c(b) = Q$ are the endpoints of $\mathcal{C}$. Any simple curve has two possible orientations. A curve equipped with one of these two orientations is called an oriented simple curve.

Definition 3.5 *We say that a simple curve* $\mathcal{C}$ *is closed if moreover we have* $c(a) = c(b)$*. Any simple and closed curve has two possible orientations.*

Definition 3.6 *Let* $\mathcal{C}$ *be a simple orientated curve (eventually closed). Then we set*

$$\int_{\mathcal{C}} F.ds = \int_c F.ds$$

where c *is any parametrization but adapted and preserving the orientation of* $\mathcal{C}$*.*

Be careful: we need to check each time that the parametrization is adapted and preserves the orientation.

Example 3.7 *Let* $c(t) = (\cos t, \sin t, 0)$ *and* $p(t) = (\cos 2t, \sin 2t, 0)$, $0 \leq t \leq 2\pi$*, and* $F = (y, 0, 0)$*. Then we find that*

$$\int_c F.ds = -\pi \ \textit{but} \ \int_p F.ds = -2\pi$$

Note that $Im\ c = Im\ p$*.* p *is not injective.*

Proposition 3.1 *Let* $\mathcal{C}^-$ *the same curve as* $\mathcal{C}$ *but with the opposite orientation. Then*

$$\int_{\mathcal{C}} F.ds = -\int_{\mathcal{C}^-} F.ds$$

Remark 3.1 *We may generalize all these facts to "sums" of oriented maps.*

4 Exercices of this Chapter

1. Compute the **path** integrals $\int_c f(x, y, z)ds$ for each case:

   (a) $f(x, y, z) = y$, $c(t) = (0, 0, t)$, $0 \leq t \leq 1$.

(b) $f(x,y,z) = x + y + z$, $c(t) = (\sin t, \cos t, t)$, $t \in [0, 2\pi]$.

(c) $f(x,y,z) = e^{\sqrt{z}}$, $c(t) = (1, 2, t^2)$, $t \in [0, 1]$.

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function. Show that the **path** integral of f along a given path c in polar coordinates by $r = r(\theta)$, $\theta_1 \leq \theta \leq \theta_2$ is:

$$\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + (\frac{dr}{d\theta})^2}\ d\theta$$

3. Let$f : [a, b] \to \mathbb{R}$ be a C^1 function (by pieces). We call length of the associated curve to f, denoted by $L(f)$, the length of the path $t \to (t, f(t))$, $t \in [a, b]$.

(a) Show that

$$L(f) = \int_a^b \sqrt{1 + (f'(x))^2} dx$$

(b) End the computations when $f(x) = \ln x$, $a = 1$, $b = 2$.

4. Let $F(x,y,z) = (x, y, z)$. Compute the **line** integral of F in each case:

(a) $c(t) = (t, t, t), t \in [0, 1]$.

(b) $c(t) = (\cos t, \sin t, 0), t \in [0, 2\pi]$.

(c) $c(t) = (\sin t, 0, \cos t), t \in [0, 2\pi]$.

(d) $c(t) = (t^2, 3t, 2t^3), t \in [-1, 2]$

5. Compte each of the following **line** integrals:

(a) $\int_c xdy - ydx$, $c(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

(b) $\int_c xdx + ydy$, $c(t) = (\cos \pi t, \sin \pi t)$, $t \in [0, 2]$.

(c) $\int_c yzdx + xzdy + xydz$, where c is made of the segments from $(1, 0, 0)$ to $(0, 1, 0)$ to $(0, 0, 1)$.

6. Let c be a sufficiently regular path.

(a) Assume that F is orthogonal to $c'(t)$ at $c(t)$. Show that

$$\int_c F.ds = 0$$

(b) Assume that $F(c(t)) = \lambda(t)c'(t)$,wher $\lambda(t) > 0$. Show that

$$\int_c F.ds = \int_c \parallel F \parallel ds$$