Lecture 2

Linear Regression

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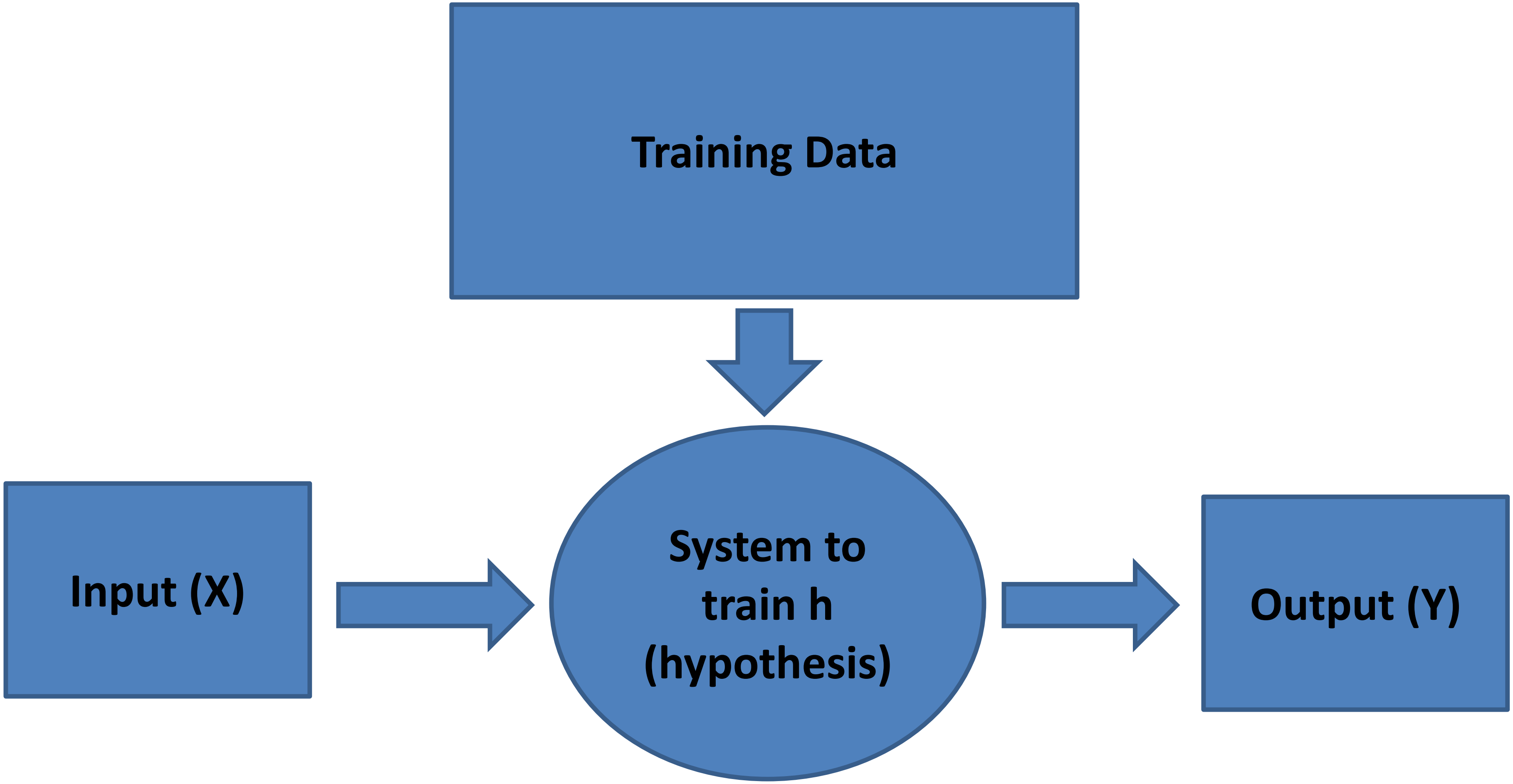
Outline

- Linear Regression
- Gradient Descent?
- Multi Feature Representation
- Questions and Answers

Review of Machine Learning



For each input *x*, output *y* = *h* (*x*)

Linear Regression

Size (m^2)	Price (billion VND)
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

$(x^{(2)},y^{(2)})$

$(x^{(5)},y^{(5)})$

Table 1: Training data of housing price in Hanoi

In Supervised Learning, each training data consists of 2 elements: input x (features) and output y (response)

Notation

(x,y) : One training data

$(x^{(i)},y^{(i)})$: the i^{th} training data

Plotting of training data

Linear Regression: Assume that the output ***y*** is a linear function of input ***x***

$$y=h(x)=a*x+b$$

Linear Regression

Objective:

Learning the function *y=h(x)=a*x+b*, such as it returns the minimize error- cost function for the training data (optimization problem)

Find the coefficient *a,b* to minimize the cost function

Cost Function:

The error for training data:

Size(m²)	Price (b.VND)
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

$$e^{(1)} = \frac{1}{2}(h(x^{(1)}) - y^{(1)})^2 = \frac{1}{2}(30a + b - 2.5)^2$$

$$e^{(2)} = \frac{1}{2}(h(x^{(2)}) - y^{(2)})^2 = \frac{1}{2}(43a + b - 3.4)^2$$

.

.

.

$$e^{(m)} = \frac{1}{2}(h(x^{(m)}) - y^{(m)})^2 = \frac{1}{2}(x^{(m)}a + b - y^{(m)})^2$$

The cost function is define as:

$$E = \frac{1}{m}(e^{(1)} + e^{(2)} + \ldots + e^{(m)}) = \frac{1}{m}\sum_{i=1}^{m} e^{(i)} = \frac{1}{2m}\sum_{i=1}^{m}(h(x^{(i)}) - y^{(i)})^2$$

$$E = \frac{1}{2m}\sum_{i=1}^{m}(ax^{(i)} + b - y^{(i)})^2$$

Gradient Descent

Objective:

Use the gradient function to find a minimum of a function

$$E = \frac{1}{2m}\sum_{i=1}^{m}(ax^{(i)} + b - y^{(i)})^2$$

Note that E is a function of a and b, we have only 2 variables a and b.

Idea:

Choose random number for a and b, the algorithm is implemented in many steps. At each step, modify a and b such s the cost function is reduced

$$a := a - \alpha \frac{\partial}{\partial_a} E(a) \qquad b := b - \alpha \frac{\partial}{\partial_b} E(b)$$

Gradient Descent

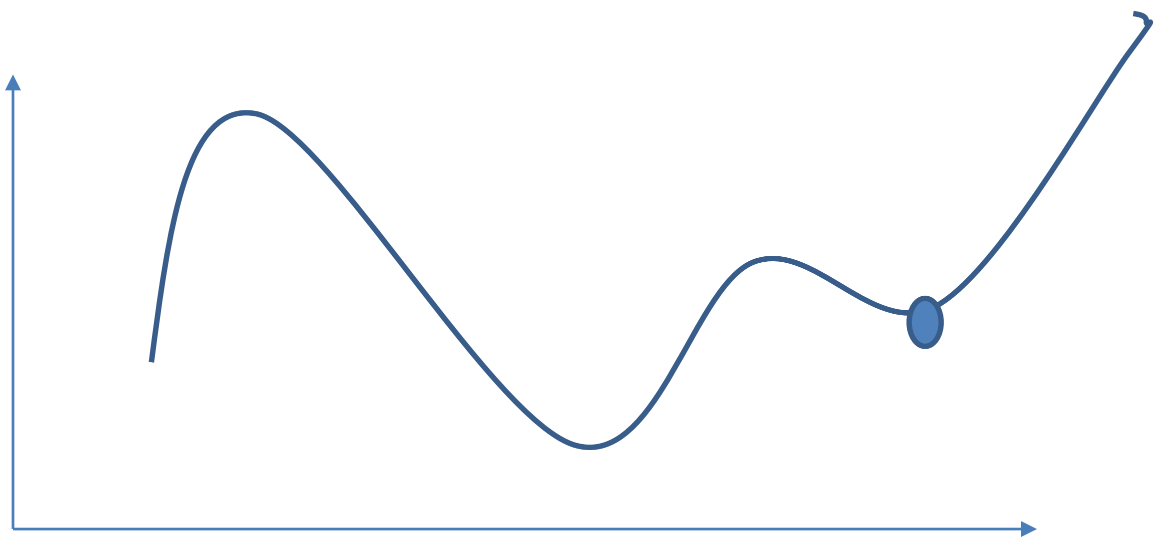
A demonstration and explanation of Gradient Descent algorithm can be found at the following website:

http://www.onmyphd.com/?p=gradient.descent

Gradient Descent

Suppose that (x0,y0) is a local minimum of the cost function, what will 1 iteration of gradient descent do?

1. Leave x0 unchanged
2. Change x0 in random direction
3. Move x0 toward the global minimum
4. Decrease x0



Gradient Descent

Calculate the derivations of the cost function: $\frac{\partial}{\partial_a}E(a)$ $\frac{\partial}{\partial_b}E(b)$

$$E = \frac{1}{2m}\sum_{i=1}^{m}(ax^{(i)} + b - y^{(i)})^2$$

Given the following formula:

$$\frac{\partial}{\partial_x}(x^2) = 2x$$

$$\frac{\partial}{\partial_x}(f(x)^2) = 2f(x).\frac{\partial}{\partial_x}f(x)$$

Gradient Descent

$$\frac{\partial}{\partial_a} E(a) = \frac{1}{m} \sum_{i=1}^{m} (ax^{(i)} + b - y^{(i)}).x^{(i)}$$

$$\frac{\partial}{\partial_b} E(b) = \frac{1}{m} \sum_{i=1}^{m} (ax^{(i)} + b - y^{(i)})$$

Gradient Descent

Exercise:

Starting at a=0 and b= 0, α=0.01, what is the cost function?

Calculate the value of a and b after first iteration (first step). Confirm if the cost function is reduced or not?

Size(m²)	*Price (b.VND)*
30	2.5
43	3.4
25	1.8
51	4.5
40	3.2
20	1.6

Gradient Descent

Batch and Stochastic Gradient Descent

Batch gradient Descent:

Compute the Gradient Descent using the whole data set

Stochastic Gradient Descent:

Compute the Gradient Descent using 1 training example at a time

- Randomly reorder the training data
- Use $(x^{(1)},y^{(1)})$ to calculate the gradient descent in order to update a,b
- Use $(x^{(2)},y^{(2)})$ to update
- ……

Gradient Descent

Mini-Batch Gradient Descent

Compute the Gradient Descent for t example at a time (1<t<m)

Example:

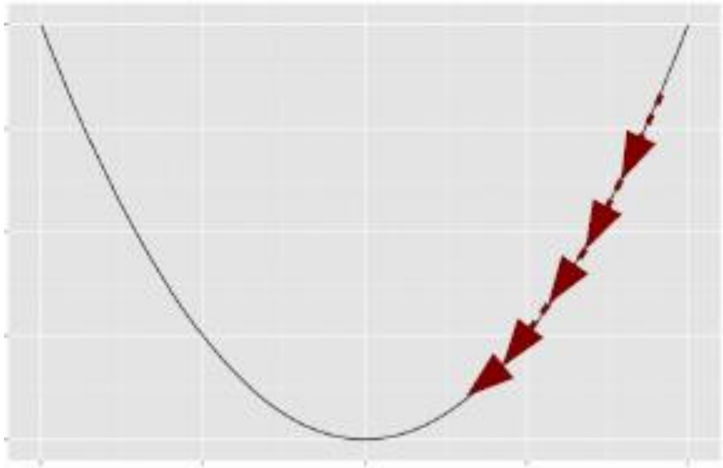
We have 1000 training data.

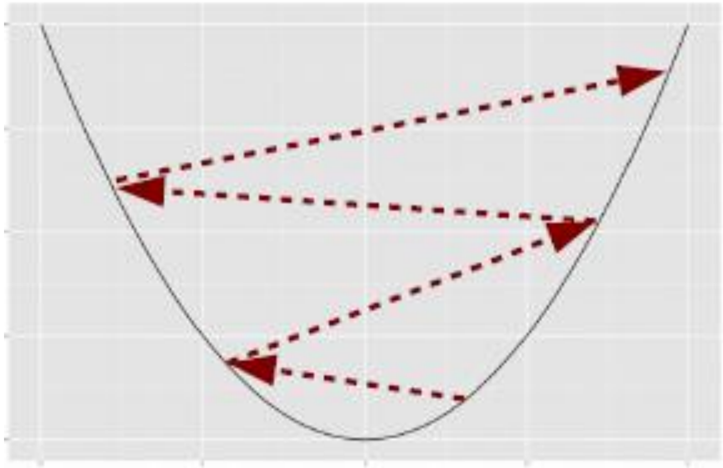
Step 1: Update the coefficient using 10 data 1-10

Step 2: Update the coefficient using 10 data 11-20

….

Gradient Descent





Big value of α may lead to the incensement of cost function and not convergence

⇒ Gradient Descent works if the cost function decrease at each step

Gradient Descent

Convergence:

How to know if a function is converged or not?

- Cost function is smaller than a predefined threshold
- After a big enough number of step
- Cost function decreased less than a predefine threshold

Gradient Descent

Summarization:

1. **Calculate the cost function**
2. **Select random value for coefficient a,b**
3. **Step by step modify a, b such as the cost function is decreased**

While (not converged)

do

$$a := a - \frac{\partial}{\partial_a} E(a) \qquad\qquad b := b - \frac{\partial}{\partial_b} E(b)$$

Multiple Input Representation

Example:

Consider the same example, but with more inputs

Size (m^2)	N^0 of floors	N^0 of rooms	Price (billion VND)
30	3	6	2.5
43	4	8	3.4
25	2	3	1.8
51	4	9	4.5
40	3	5	3.2
20	1	2	1.6

$x^{(i)}$: the input of i^{th} training data

$x_j^{(i)}$: the component j of i^{th} training data

Multiple Input Representation

Matrix representation:

$$y = h(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_m x_m$$

$$x = \begin{bmatrix} 1 \\ x_1 \\ \ldots \\ x_m \end{bmatrix} \qquad \theta = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \ldots \\ \theta_m \end{bmatrix}$$

$$h(x) = [\theta_0\ \theta_1\ \theta_2 \ldots \theta_m] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \ldots \\ x_m \end{bmatrix} = \theta^T x$$

Multiple Input Representation

Cost Function:

$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m}\sum_{i=1}^{m}(\theta^T x^{(i)} - y^{(i)})^2$$

Multiple Input Representation

<u>Gradient Descent</u>

Start with random value of θ, step by step modify θ in order to decrease the cost function

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial_{\theta_j}} E(\theta)$$

$$\frac{\partial}{\partial_{\theta_j}} E(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial_{\theta_j}} (\theta^T x^{(i)} - y^{(i)})^2$$

$$= \frac{1}{m} \sum_{i=1}^{m} (\theta^T x^{(i)} - y^{(i)}) x_j^{(i)}$$

Multiple Input Representation

Gradient Descent

Start with random value of θ, step by step modify θ in order to decrease the cost function

$$\theta_j := \theta_j - \alpha \frac{\partial}{\partial_{\theta_j}} E(\theta)$$

$$\frac{\partial}{\partial_{\theta_j}} E(\theta) = \frac{1}{2m} \sum_{i=1}^{m} \frac{\partial}{\partial_{\theta_j}} (\theta^T x^{(i)} - y^{(i)})^2$$

$$= \frac{1}{m} \sum_{i=1}^{m} (\theta^T x^{(i)} - y^{(i)}) x_j^{(i)}$$

Normal Equations

Linear Regression:

Minimize the value of the cost function:

$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m}\sum_{i=1}^{m}(\theta^T x^{(i)} - y^{(i)})^2$$

Normal Equations:

Solve the following equation to find out the optimized value of θ

$$\frac{\partial}{\partial_\theta} E(\theta) = 0$$

Normal Equations

<u>Linear Regression:</u>

Minimize the value of the cost function:

$$E(\theta) = \frac{1}{2m}\sum_{i=1}^{m}(h_\theta(x^{(i)}) - y^{(i)})^2 = \frac{1}{2m}\sum_{i=1}^{m}(\theta^T x^{(i)} - y^{(i)})^2$$

<u>Normal Equations:</u>

Solve the following equation to find out the optimized value of θ

$$\frac{\partial}{\partial_\theta}E(\theta) = 0 \qquad \forall\ j \in (0,1,\ldots,n), \frac{\partial}{\partial_{\theta_j}}E(\theta) = 0$$

Normal Equations

Solution:

Given a training set of m training example, each contain n inputs, we have the matrix X (m,n+1) of inputs and vector of output Y

$$X = \begin{bmatrix} x_0^{(1)} & x_1^{(1)} & \dots & x_n^{(1)} \\ x_0^{(2)} & x_1^{(2)} & \dots & x_n^{(2)} \\ & & \dots & \\ x_0^{(m)} & x_1^{(m)} & \dots & x_n^{(m)} \end{bmatrix} = \begin{bmatrix} (x^{(1)})^T \\ (x^{(2)})^T \\ \dots \\ (x^{(m)})^T \end{bmatrix} \qquad Y = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \dots \\ y^{(m)} \end{bmatrix}$$

Solution of normal equations is:

$$\theta = (X^T X)^{-1} X^T Y$$

Homework

Write a program in (Matlab/C++)to implement gradient descent algorithm for the following training data with different learning method: batch learning, stochastic and mini-batch, normal equation. Send your code and report on what do you see from the result to my email before 20 October 2016

Size (m^2)	N^0 of floors	N^0 of rooms	Price (billion VND)
30	3	6	*2.5*
43	4	8	*3.4*
25	2	3	*1.8*
51	4	9	*4.5*
40	3	5	*3.2*
20	1	2	*1.6*

Polynomial Regression

Polynomial Regression

Output is an polynomial function of the input

For example

$$h(x) = \theta_0 + \theta_1 x + \theta_2 x^2 + \ldots + \theta_n x^n$$

Assume

$$x_1 = x$$

$$x_2 = x^2$$

$$\ldots$$

$$x_n = x^n$$

$$h(x) = \theta_0 + \theta_1 x_1 + \theta_2 x_2 + \ldots + \theta_n x_n$$

Linear Regression

References

http://openclassroom.stanford.edu/MainFolder/VideoPage.php?course=MachineLearning&video=02.4-LinearRegressionI-GradientDescent&speed=100

Feature Rescale

Objective: Scale all features to the same scale, in order to have easier computation

Popular scale : [0,1],[-0.5,0.5]

x=x/max (x)

c= mean (x) x=x-c/max(x) =>