Mathematics for Physicists
Lecture 2
Surface Integrals

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General facts

We have

Proposition 1.1

Let $g : \mathbb{R}^3 \to \mathbb{R}$ of class C^1 and let S the level surface associated to g and $k \in \mathbb{R}$, that is

$$S = \{(x, y, z) \in \mathbb{R}^3, \text{ s.t. } g(x, y, z) = k\}$$

Assume that $S \neq \emptyset$. Let $(x_0, y_0, z_0) \in S$ be fixed. Then the vector $\nabla g(x_0, y_0, z_0)$ is normal to S at point (x_0, y_0, z_0), in the following sense : for any path $c : [a, b] \to S$, with $c(0) = (x_0, y_0, z_0)$, of classe C^1, if we let $v = c'(0)$, that is the tangent vector to c at 0, then $\nabla g(x_0, y_0, z_0)$ and v are orthogonal that is $\nabla f(x_0, y_0, z_0).v = 0$.

Proof is easy.

One can then introduce the notion of the tangent plane to S at a fixed point of S as being the plane going through this point and normal to this vector. We have

Definition 1.1

Let $g : \mathbb{R}^3 \to \mathbb{R}$, C^1. Let $k \in \mathbb{R}$ and $S = \{g(x, y, z) = k\}$ the level surface of height k, associated to g. Assume that $S \neq \emptyset$. Then for all point $(a, b, c) \in S$, we define the tangent plane to S at point (a, b, c) as being the plane with cartesian equation

$$\nabla g(a, b, c).(x - a, y - b, z - c) = 0$$

Parametrized surfaces

There exists surfaces of $\mathbb{R}^3$ which are not graphs of functions. Notion of parametrized surfaces which include in particular the case of functions graphs.

Definition 1.2

A parametrized surface of $\mathbb{R}^3$ is a map $\Phi : D \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$. The corresponding surface is $S = \Phi(D)$.

Thus, if we denote by x, y, z the component functions of Φ, we have

$$\Phi(u, v) = (x(u, v), y(u, v), z(u, v))$$

Thus, in order to describe $\mathcal{S}$, we need two variables u and v : we can say that the dimension of $\mathcal{S}$ is two.
One can also say that Φ is a parametrization of $\mathcal{S}$.
Note that this very similar to the previous chapter on paths.

Careful : make the difference between a parametrized surface, which is a map from $\mathbb{R}^2$ to $\mathbb{R}^3$ and the corresponding "surface" which is a set of point in $\mathbb{R}^3$.

Definition of a normal vector or a tangent plane to $\mathcal{S}$ as in the case of level surfaces.
Let (u_0, v_0) be a fixed point in D. Fix $u = u_0$ and consider the path $t \in \mathbb{R} \to \Phi(u_0, t)$.
Its image, that is the associated curve, is contained in $\mathcal{S}$. A tangent vector to this path, at point $\Phi(u_0, v_0)$ is

$$T_v = (\partial_v x(u_0, v_0), \partial_v y(u_0, v_0), \partial_v z(u_0, v_0))$$

Similarly, we introduce the vector

$$T_u = (\partial_u x(u_0, v_0), \partial_u y(u_0, v_0), \partial_u z(u_0, v_0))$$

These two vectors T_u and T_v are tangent to two curves of $\mathcal{S}$ at point $\Phi(u_0, v_0)$.
This suggests to say that a normal vector to $\mathcal{S}$ at point $\Phi(u_0, v_0)$ should be $T_u \wedge T_v$, unless it is zero.

Definition 1.3

We say $(\mathcal{S}, \Phi)$ *is regular at* $\Phi(u_0, v_0)$ *if* $T_u \wedge T_v \neq 0$ *at* (u_0, v_0)*.*
We say that this surface is regular if is regular all of its points.
In these cases, we say that $\vec{n} = T_u \wedge T_v$ *is normal to* $\mathcal{S}$ *at point* $\Phi(u_0, v_0)$*. And we call tangent plane to* $\mathcal{S}$ *at point* $\Phi(u_0, v_0)$ *the plane with cartesian equation*

$$\vec{n}.(x - a, y - b, z - c) = 0$$

where $(a, b, c) = \Phi(u_0, v_0)$*.*

Example 1.1

$x = u \cos v, y = u \sin v z = u^2 + v^2$*. Find the tangent plane at* $\Phi(1, 0)$*.*

Special case : a surface $\mathcal{S}$ given by the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$, C^1 and defined on a subset D of $\mathbb{R}^2$.
One can check that a classical parametrization of $\mathcal{S}$ is given by :

$$\Phi : (u, v) \in D \to (x, y, z)$$

with

$$x = u, y = v, z = g(u, v)$$

We find

$$T_u = (1, 0, \partial_u g(u, v), T_v = (0, 1, \partial_v g(u, v))$$

Thus a normal vector is given by

$$\vec{n}(u, v) = T_u \wedge T_v = (-\partial_u g, -\partial_v g, 1) \neq 0$$

Note that this parametrization Φ is regular. Note also that vector $\vec{n}$ always points in the upper direction.

Area of a surface

Let be given a (good) parametrized surface with a good initial set D :
We want to define the notion of area of the surface $(\mathcal{S}, \Phi)$:

Definition 1.4

The area of $(\mathcal{S}, \Phi)$ is the positive number given by :

$$area(\mathcal{S}, \Phi) = \int\int_D \parallel T_u \wedge T_v \parallel dudv$$

Note that letter Φ appears in the above formula. We shall soon see that this area does not depend on the change of parametrization of $\mathcal{S}$. Note that we have

$$(area(\mathcal{S}, \Phi) = \int\int_D \sqrt{\mid \frac{\partial(x, y)}{\partial(u, v)} \mid^2 + \mid \frac{\partial(y, z)}{\partial(u, v)} \mid^2 + \mid \frac{\partial(x, z)}{\partial(u, v)} \mid^2} dudv$$

Explanations :
Assume that D is a rectangle of $\mathbb{R}^2$.
Let a partition of order n of D in small rectangles denoted by R_{ij}.
Denote the four points of each of these rectangles by (u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1}) et (u_{i+1}, v_{j+1}), où $0 \leq i \leq n-1$ et $0 \leq j \leq n-1$.
Set T_{u_i} et T_{v_j} for the values of T_u and T_v at points (u_i, v_j).
Vectors $\Delta u T_{u_i}$ and ΔT_{v_j} are tangent to $\mathcal{S}$ at point $\Phi(u_i, v_j) = (x_{ij}, y_{ij}, z_{ij})$, with $\Delta u = u_{i+1} - u_i$ et $\Delta v = v_{j+1} - v_j$.
These two vectors form a parallelogram denoted by P_{ij} included in the tangent plane to $\mathcal{S}$. If n is large enough, we have a kind of covering of $\mathcal{S}$ by these P_{ij}.
When n is large enough, we have

$$area(P_{ij}) \simeq area(\Phi(R_{ij}))$$

As

$$area(P_{ij}) \simeq \| \Delta u T_{U_i} \wedge \Delta v T_{v_j} \| = \| T_{U_i} \wedge T_{v_j} \| \Delta u \Delta v$$

we deduce by summing that the cover made by the P_{ij} is

$$A_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \; aire(P_{ij}) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \parallel T_{U_i} \wedge T_{v_j} \parallel \Delta u \Delta v$$

This is a Riemann summation and then we can get the previous definition.

Example 1.2

Let $\mathcal{S}$ be the cone whose one possible parametrization is given by $D = [0, 2\pi]_\theta \times [0, 1]_r$ and

$$\Phi : (r, \theta) \to (x, y, z)$$

with

$$x = r\cos\theta, y = r\sin\theta, z = r$$

We find that area$(\mathcal{S}, \Phi) = \sqrt{2}\pi$.

Example 1.3

Area of $\mathcal{S}$ (helicoidal surface) with parametrization given by $D = [0, 2\pi]_\theta \times [0, 1]_r$ et

$$\Phi : (r, \theta) \to (x, y, z)$$

with

$$x = r\cos\theta, y = r\sin\theta, z = \theta$$

Case of a surface given by a graph of a function.

$$area(\mathcal{S}, g) = \int\int_D \sqrt{(\partial_u f)^2 + (\partial_v f)^2 + 1} dudv$$

Particular case : area of surface obtained by revolution of the graph of $u = f(x)$ around x axis ; then

$$aire = 2\pi \int_a^b (\mid f(x) \mid \sqrt{1 + [f'(x)]^2})dx$$

If the revolution is around y axis, we get

$$area = 2\pi \int_a^b (\mid x \mid \sqrt{1 + [f'(x)]^2})dx$$

For the first formula : introduce the parametrization of $\mathcal{S}$ given by

$$x = u, y = f(u)\cos v, z = f(u)\sin v$$

on D defined by $a \leq u \leq b$ et $0 \leq v \leq 2\pi$.
For fixed u, $(u, f(u)\cos v, f(u)\sin v)$ moves along a circle of radius $\mid f(u) \mid$ centered at $(u, 0, 0)$. Then

$$\frac{\partial(x,y)}{\partial(u,v)} = -f(u)\sin v, \frac{\partial(y,z)}{\partial(u,v)} = f(u)f'(u), \frac{\partial(x,z)}{\partial(u,v)} = f(u)\cos v$$

Scalar functions integrals over surfaces

Let $(\mathcal{S}, \Phi)$ be a surface parametrized by

$$\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3,\ \Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

Definition 1.5

Let $f : S \to \mathbb{R}$. Then we set the definition

$$\int\int_\Phi f(x,y,z)dS = \int\int_\Phi f dS \equiv \int\int_D f(\Phi(u,v)) \parallel T_u \wedge T_v \parallel dudv$$

or equivalently

$$\int\int_\Phi f dS = \int\int_D f(x(u,v), y(u,v), z(u,v)) \times$$

$$\sqrt{(\frac{\partial(x,y)}{\partial(u,v)})^2 + (\frac{\partial(y,z)}{\partial(u,v)})^2 + (\frac{\partial(x,z)}{\partial(u,v)})^2} dudv$$

Example 1.4

In the helicoidal case and if $f(x,y,z) = \sqrt{x^2+y^2+1}$, *we get*

$$\int\int_{\Phi} f = \frac{8}{3}\pi$$

Example 1.5

If S *is the graph of a function* g C^1, *then*

$$\int\int_{g} fdS = \int\int_{D} f(x,y,g(x,y))\sqrt{1+(\partial_u g)^2+(\partial_v g)^2}ddxdy$$

Surface integrals of vector (valued) functions

Definition 1.6

Let F be a vector field, defined on $\mathcal{S}$, a parametrized surface by Φ. Then the surface integral of F on Φ, or the flux of F across Φ is defined by

$$\int\int_{\Phi} F.dS = \int\int_{D} F.(T_u \wedge T_v)dudv$$

Example 1.6

IF $D : \{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi\}$ and Φ is given by

$$x = \cos\theta \sin\phi, y = \sin\theta\cos\phi, z = \cos\phi$$

then S is the unit sphere of $\mathbb{R}^3$; if we introduce the vector field $\vec{r} = (x, y, z)$, then $\oint\int_{\Phi} r.dS = -4\pi$.

Definition of "orientation"

Definition 1.7

A oriented surface is a surface with "two sides", where one side could be called the positive or exterior one, and the other side the negative side or the interior one. This is so that at each point (x, y, z) of this surface, there exists two unit normal vectors n_1 and n_2, $n_1(x, y, z)$ and $n_2(x, y, z)$ pointing in opposite directions, n_1 pointing towards the positive side, while n_2 points towards the negative side, in a continuous way. Thus, to specify a side of $\mathcal{S}$, at all point of $\mathcal{S}$, we choose a unit normal vector $\vec{n}$ always pointing to the exterior.

Remark 1.1

This definition rests on the fact that we should be able to talk about the "two sides" of the surface $\mathcal{S}$.

Let $\Phi : D \to \mathbb{R}^3$ be a parametrization of an oriented surface $\mathcal{S}$.
Assume that $\mathcal{S}$ is regular at $\Phi(u_0, v_0)$, $(u_0, v_0) \in D$.
In that case, the vector $T_u \wedge T_v(u_0, v_0) \neq 0$,
$\| T_u \wedge T_v(u_0, v_0) \| \neq 0$, and thus the vector $T_u \wedge T_v(u_0, v_0)$ is normal to $\mathcal{S}$ at point $\Phi(u_0, v_0)$.
We obtain an unit normal vector if we consider the vector $\frac{T_u \wedge T_v(u_0, v_0)}{\| T_u \wedge T_v(u_0, v_0) \|}$.
As the surface $\mathcal{S}$ is oriented, we have done the choice of a normal vector field $\vec{n}$ always directed towards the same side, called the positive one. Thus we have

$$\frac{T_u \wedge T_v(u_0, v_0)}{\| T_u \wedge T_v(u_0, v_0) \|} = \mp \vec{n}(\Phi(u_0, v_0))$$

Definition 1.8

With the above definitions, we say that Φ preserves the orientation of $\mathcal{S}$, if we have always the $+$ sign in the above equality ; that is if the vector $T_u \wedge T_v$ always points towards the exterior (which is already fixed as we have an oriented surface).
If, on the other hand, $T_u \wedge T_v$ always points towards the interior, we say that Φ reverses the orientation, that is we have always the $-$ sign in the above equality.

Example 1.7

Consider the unit sphere $\mathcal{S} : x^2 + y^2 + z^2 = 1$. Choose the exterior side of $\mathcal{S}$. Let Φ be the parametrization of $\mathcal{S}$ given by $D = \{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$ and

$$x = \cos\theta \sin\phi, y = \sin\theta \sin\phi, z = \cos\phi$$

We find $T_\theta \wedge T_\phi = -r \sin\phi$. As $\sin\phi \leq 0$, $T_\theta \wedge T_\phi$ points always to the interior. Thus Φ reverses the orientation.

Example 1.8

Let $\mathcal{S}$ be the graph of a function g. A normal vector at point (x,y,z) to $\mathcal{S}$ is

$$T_u \wedge T_v = (-\partial_u g, -\partial_v g, 1)$$

We get two unit normal vectors by setting

$$\vec{n} = \mp[(\partial_u g)^2 + (\partial_v g)^2 + 1]^{\frac{1}{2}}(-\partial_u g, -\partial_v g, 1)$$

The third component is always positive (if we choose the $+$ sign). Thus we can always choose the orientation of $\mathcal{S}$ by taking as the $+$ side, the side where $\vec{n}$ points to.
In that case, Φ preserves the orientation.

Theorem 1.1

Let $\mathcal{S}$ be an oriented surface, and F a continuous vector field defined on $\mathcal{S}$.
1) If Φ_1 and Φ_2 are two parametrizations preserving the orientations of $\mathcal{S}$, then

$$\int\int_{\Phi_1} F.dS = \int\int_{\Phi_2} F.dS$$

2) IF Φ_1 et Φ_2 are two parametrizations reversing the orientation of $\mathcal{S}$, then

$$\int\int_{\Phi_1} F.dS = -\int\int_{\Phi_2} F.dS$$

Note that for scalar functions $f : \mathbb{R}^3 \to \mathbb{R}$, we have always

$$\int\int_{\Phi_1} fdS = \int\int_{\Phi_2} fdS$$

Definition 1.9

1) Case of integrals of scalar functions. Let $\mathcal{S}$ be a parametrized surface. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuous function. Then, by definition, we set

$$\int\int_{\mathcal{S}} f dS = \int\int_{\Phi} f dS$$

where Φ is any but "good" parametrization of $\mathcal{S}$.

2) Case of integrals of vector functions. Let $\mathcal{S}$ be an oriented parametrized surface. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then we set

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\Phi} F.dS$$

where Φ is any but good paramatrization of $\mathcal{S}$, preserving the orientation of $\mathcal{S}$. Similarly, we set

$$\int\int_{\mathcal{S}^-} F.dS = \int\int_{\Phi} F.dS$$

wher Φ is any but good parametrization of $\mathcal{S}$, reversing the orientation de $\mathcal{S}$.

Note that

$$\int\int_{\mathcal{S}^+} F.dS = -\int\int_{\mathcal{S}^-} F.dS$$

The quantity $\int\int_{\mathcal{S}^+} F.dS$ is called the flux of F across the positively oriented surface $\mathcal{S}$.

Final remark.

Let $\mathcal{S}$ be a regular and oriented surface, with Φ a parametrization preserving the orientation. In particular, $n = \frac{T_u \wedge T_v}{\| T_u \wedge T_v \|}$ is the unit normal vector pointing to the exterior of $\mathcal{S}$ (positive side of $\mathcal{S}$).

We get

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\Phi} F.dS = \int\int_D F.(T_u \wedge T_v)dudv =$$

$$= \int\int_D (\frac{T_u \wedge T_v}{\| T_u \wedge T_v \|}) \| T_u \wedge T_v \| dudv =$$

$$= \int\int_D (F.n) \| T_u \wedge T_v \| dudv = \int\int_{\mathcal{S}} (F.n)dS$$

Thus

Proposition 1.2

With the above notations, we have

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\mathcal{S}} (F.n)dS$$

Careful : the first integral is a flux, that is a surface integral of a vector functions, here F, while the second integral is the surface integral of a sclar function, here $F.n$.

Applied to the case of a surface $\mathcal{S}$ given by the graph of a function g, we get

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{D} [F_1(-\partial_u g) + F_2(-\partial_v g) + F_3] dudv$$

where $F = (F_1, F_2, F_3)$ are the components of the vector field.