Chapter 1

Surface integrals

1.1 Generalities

We known from basic mathematics the notion of a graph of a function, in $\mathbb{R}^2$ or in $\mathbb{R}^3$, in which case, this graph is one example of a surface.

We want to consider more general cases, and in particular introduce the notion of a normal vector to a surface.

For us, and for now, a surface will be a subset of $\mathbb{R}^3$, which could be described either as the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$, or as a level surface associated to a function $g : \mathbb{R}^3 \to \mathbb{R}$. We shall see later on a third example of a surface.

It is clear that the graph associated to a function $h : \mathbb{R}^2 \to \mathbb{R}$ is also a level surface associated to a function $g : \mathbb{R}^3 \to \mathbb{R}$.

So for now, we can work with level surface, whose definition is recalled below.

We have the first result:

Proposition 1.1.1 *Let $g : \mathbb{R}^3 \to \mathbb{R}$ be of class C^1 and let S be the level surface associated with g and to $k \in \mathbb{R}$, that is*

$$S = \{(x, y, z) \in \mathbb{R}^3, \text{ tel que } g(x, y, z) = k\}$$

We assume that S is not empty. Let $(x_0, y_0, z_0) \in S$ be fixed. Then the vector $\nabla g(x_0, y_0, z_0)$ is normal to S at point (x_0, y_0, z_0), in the following sense: for any path $c : [a, b] \to S$, with $c(0) = (x_0, y_0, z_0)$, of class C^1, if we set $v = c'(0)$, that is the tangent vector to c at 0, then $\nabla g(x_0, y_0, z_0)$ and v are orthogonal, that is $\nabla f(x_0, y_0, z_0).v = 0$.

proof: Let c be such a path. Thus $c(t) \in S$, for all $t \in [a, b]$. Thus we have $g(c(t)) = k$, $\forall t \in [a, b]$. Define $h(t) = f(c(t))$, $\forall t \in [a, b]$. The function $h : [a, b] \to \mathbb{R}$ is C^1 and thus is constant. Thus $h'(t) = 0$, $\forall t \in [a,b]$. But as $h'(t) = \nabla g(c(t)).c'(t)$, we obtain $\nabla g(c(0)).c'(0) = 0$.

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In conclusion, this proposition gives us the definition of a normal vector to S at a point of S, but also an important example of such a vector.
Of course, any vector parallel to this one will be also normal to the surface S.
We can now say that the tangent plane to S will be the plane through that point and orthogonal to this normal vector.

Définition 1.1.1 *Let $g : \mathbb{R}^3 \to \mathbb{R}$, C^1. Let $k \in \mathbb{R}$ and $S = \{g(x,y,z) = k\}$ the level surface at level k associated with g. We asume that S is not empty. Then for any point $(a,b,c) \in S$, we define the tangent plane to S at point (a,b,c) as the plan with cartesian equation given*

$$\nabla g(a,b,c).(x-a, y-b, z-c) = 0$$

1.2 Parametrized surfaces

We know that a basic example of a surface is given by graphs of functions. Another example is given by level surfaces associated with functions from $\mathbb{R}^3$ into $\mathbb{R}$. Such surfaces could as well not be associated with graphs.

We shall now introduce the notion of a parametrized surface which includes the particular case of graphs of functions.

Définition 1.2.1 *A parametrized surface of $\mathbb{R}^3$ is a map $\Phi : D \to \mathbb{R}^3$, where $D \subset \mathbb{R}^2$. The corresponding (geometric) surface is $\mathcal{S} = \Phi(D)$.*

Thus, denoting by x, y, z the components functions of Φ, we have

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

Thus we see that to describe $\mathcal{S}$, we need two variables u and v: we can thus say that the dimension of $\mathcal{S}$ is two.
If Φ is C^0 or C^1 or ..., we then say that $\mathcal{S}$ is $C^0 1$ or C^1 or ...
Finally, we say that Φ is a parametrization of $\mathcal{S}$. Note the similarity with the curves framework.
In particular, we should make the difference between a parametrized surface, which is map from $\mathbb{R}^2$ into $\mathbb{R}^3$ and the corresponding surface, which is a set of points in $\mathbb{R}^3$.

In the following, we want to give a definition of the notion of normal vector or tangent plane to $\mathcal{S}$ as in the case of level surfaces.
We assume in the following that Φ is at least C^1. Let (u_0, v_0) be a fixed point in D. Fix $u = u_0$ and consider the map $t \in \mathbb{R} \to \Phi(u_0, t)$. This is a path in $\mathbb{R}^3$. Its image, that is the associated curve, is contained in $\mathcal{S}$. We have seen that a tangent vector to this path, at point $\Phi(u_0, v_0)$ was

$$T_v = (\partial_v x(u_0, v_0), \partial_v y(u_0, v_0), \partial_v z(u_0, v_0))$$

Similarly, we introduce the vector

$$T_u = (\partial_u x(u_0, v_0), \partial_u y(u_0, v_0), \partial_u z(u_0, v_0))$$

We see that these two vectors T_u et T_v are tangent to two curves of $\mathcal{S}$ at point $\Phi(u_0, v_0)$. This suggests to say that a normal vector to $\mathcal{S}$ at point $\Phi(u_0, v_0)$ will be $T_u \wedge T_v$. There is a small issue, in that this vector should be non zero. This is why we set

Définition 1.2.2 *We say that* $(\mathcal{S}, \Phi)$ *is regular at* $\Phi(u_0, v_0)$ *if* $T_u \wedge T_v \neq 0$ *at* (u_0, v_0). *We say that this surface is regular it this is so at any point.*
In these cases, we say that $\vec{n} = T_u \wedge T_v$ *is normal to* $\mathcal{S}$ *at point* $\Phi(u_0, v_0)$. *Still in that case, we call the tangent plane to* $\mathcal{S}$ *at point* $\Phi(u_0, v_0)$ *the plane with cartesian equation*

$$\vec{n}.(x - a, y - b, z - c) = 0$$

where $(a, b, c) = \Phi(u_0, v_0)$.

Exemple 1.2.1 $x = u\cos v, y = u\sin v z = u^2 + v^2$. *Find the tangent plane at* $\Phi(1, 0)$.

We shall work out a particular case, which is the case of a surface $\mathcal{S}$ given by the graph of a function $g : \mathbb{R}^2 \to \mathbb{R}$, C^1 and defined on a subset D of $\mathbb{R}^2$. In that case, one can check that a classical parametrization of $\mathcal{S}$ is given by:

$$\Phi : (u, v) \in D \to (x, y, z)$$

with

$$x = u, y = v, z = g(u, v)$$

We find computing that

$$T_u = (1, 0, \partial_u g(u, v), T_v = (0, 1, \partial_v g(u, v))$$

Then a normal vector is given by

$$\vec{n}(u,v) = T_u \wedge T_v = (-\partial_u g, -\partial_v g, 1) \neq 0$$

Thus the parametrization Φ is regular at any point. Note that the vector $\vec{n}$ points always upwards.

IN THE FOLLOWING, WE SHALL ALWAYS ASSUME THAT PARAMER-TRIZATIONS ARE ALMOST INJECTIVE.

1.3 Area of a surface

Let us be given a parametrized surface such that:

$$\begin{cases} \text{- the initial set } D \text{ is an elementary subset of } \mathbb{R}^2 \\ \text{- } \Phi \text{ is almost everywhere } C^1 \\ \text{- } \mathcal{S} \text{ is almost everywhere regular} \end{cases} \tag{1.3.2}$$

We do not insist too much on the precise assumptions.

Définition 1.3.1 *We call area of* $(\mathcal{S}, \Phi)$ *the positive number given by:*

$$area(\mathcal{S}, \Phi) = \int\int_D \| T_u \wedge T_v \| \, dudv$$

Note that letter Φ appears in the notation. We shall see later on that we do not change this area if we change the parametrization of $\mathcal{S}$, under suitable assumptions.

Note immediately that

$$(area(\mathcal{S}, \Phi) = \int\int_D \sqrt{\left| \frac{\partial(x,y)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(y,z)}{\partial(u,v)} \right|^2 + \left| \frac{\partial(x,z)}{\partial(u,v)} \right|^2} dudv$$

Let us now explain where this definition comes from.
To simplify, we assume that D is a rectangle in $\mathbb{R}^2$.
Let be a subdivision of order n of D in rectangles denoted by R_{ij}.
Denote the 4 vertices of each of these rectangles by (u_i, v_j), (u_{i+1}, v_j), (u_i, v_{j+1}) and (u_{i+1}, v_{j+1}), where $0 \leq i \leq n-1$ et $0 \leq j \leq n-1$.
Define T_{u_i} and T_{v_j} as the values of T_u and T_v at points (u_i, v_j).
The vectors $\Delta u T_{u_i}$ and ΔT_{v_j} are tangent to $\mathcal{S}$ at point $\Phi(u_i, v_j) = (x_{ij}, y_{ij}, z_{ij})$, where $\Delta u = u_{i+1} - u_i$ et $\Delta v = v_{j+1} - v_j$. These two vectors form a parallogram denoted by P_{ij} included in the tangent plane to $\mathcal{S}$. This is as if we had a cover of $\mathcal{S}$ by these P_{ij}, at least when n is large enough.

Note that if n is large enough, we have

$$area(P_{ij}) \simeq area(\Phi(P_{ij}))$$

As

$$area(P_{ij}) \simeq \parallel \Delta u T_{U_i} \wedge \Delta v T_{v_j} \parallel = \parallel T_{U_i} \wedge T_{v_j} \parallel \Delta u \Delta v$$

we deduce that the area of this cover formed by the P_{ij} is

$$A_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} area(P_{ij}) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \parallel T_{U_i} \wedge T_{v_j} \parallel \Delta u \Delta v$$

We recognized a Riemann summation, and thus, we see, that when n becomes larger and larger, we shall get the formula given by the definition.

Exemple 1.3.1 *Let $\mathcal{S}$ be the cone whose parametrization is given by $D = [0, 2\pi]_\theta \times [0, 1]_r$ and*

$$\Phi : (r, \theta) \to (x, y, z)$$

with

$$x = r \cos\theta, y = r \sin\theta, z = r$$

A small computation gives that $area(\mathcal{S}, \Phi) = \sqrt{2}\pi$.

Exemple 1.3.2 *Compute the area of de $\mathcal{S}$ (helicoid) whose possible parametrization is given by $D = [0, 2\pi]_\theta \times [0, 1]_r$ and*

$$\Phi : (r, \theta) \to (x, y, z)$$

with

$$x = r \cos\theta, y = r \sin\theta, z = \theta$$

We can go back to the particular case of a surface given by the graph of a function g; then

$$area(\mathcal{S}, g) = \int \int_D \sqrt{(\partial_u f)^2 + (\partial_v f)^2 + 1} du dv$$

As a particular case, we may consider the area of a surface generated by the rotation of the graph of a function $u = f(x)$ around the x axis; then

$$area = 2\pi \int_a^b (\mid f(x) \mid \sqrt{1 + [f'(x)]^2}) dx$$

If the rotation is around the y axis, then we get

$$area = 2\pi \int_a^b (\mid x \mid \sqrt{1 + [f'(x)]^2}) dx$$

To prove the first formula, we introduce the parametrization of $\mathcal{S}$ given by

$$x = u, y = f(u)\cos v, z = f(u)\sin v$$

over D defined by $a \leq u \leq b$ and $0 \leq v \leq 2\pi$.
For fixed u, $(u, f(u)\cos v, f(u)\sin v)$ describes a circle with radius $\mid f(u) \mid$ and centered at $(u, 0, 0)$. Then we compute

$$\frac{\partial(x,y)}{\partial(u,v)} = -f(u)\sin v, \frac{\partial(y,z)}{\partial(u,v)} = f(u)f'(u), \frac{\partial(x,z)}{\partial(u,v)} = f(u)\cos v$$

1.4 Integrals of scalar functions over surfaces

Let $(\mathcal{S}, \Phi)$ be a parametrized surface

$$\Phi : D \subset \mathbb{R}^2 \to \mathbb{R}^3,\ \Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

Définition 1.4.1 *Let $f : S \to \mathbb{R}$. Then we define*

$$\int\int_\Phi f(x,y,z)dS = \int\int_\Phi f dS \equiv \int\int_D f(\Phi(u,v)) \parallel T_u \wedge T_v \parallel dudv$$

or

$$\int\int_\Phi f dS = \int\int_D f(x(u,v), y(u,v), z(u,v)) \sqrt{(\frac{\partial(x,y)}{\partial(u,v)})^2 + (\frac{\partial(y,z)}{\partial(u,v)})^2 + (\frac{\partial(x,z)}{\partial(u,v)})^2} dudv$$

Exemple 1.4.1 *In the helicoid case, and if $f(x,y,z) = \sqrt{x^2+y^2+1}$, we obtain*

$$\int\int_\Phi f = \frac{8}{3}\pi$$

Exemple 1.4.2 *If S is the graph of a function g C^1, then*

$$\int\int_g f dS = \int\int_D f(x,y,g(x,y))\sqrt{1 + (\partial_u g)^2 + (\partial_v g)^2} ddxdy$$

1.5 Integrals of vectorial functions over surfaces

Définition 1.5.1 *Let F be a vector field, defined on $\mathcal{S}$ a surface parametrized by Φ. Then the surface integral of F over Φ or the flux of F throughΦ is defined by*

$$\int\int_\Phi F.dS = \int\int_D F.(T_u \wedge T_v)dudv$$

Exemple 1.5.1 *If $D : \{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq 2\pi\}$ and Φ is given by*

$$x = \cos\theta \sin\phi, y = \sin\theta\cos\phi, z = \cos\phi$$

then S is the unit sphere of $I\!R^3$; if we introduce the vector field $\vec{r} = (x, y, z)$, then $\oint\int_\Phi r.dS = -4\pi$.

We shall now introduce the very loose definition of oriented surfaces, in order to get ride of the parametrization Φ.

Définition 1.5.2 *An oriented surface is a surface with "two sides", with a specified side being the positive or exterior one, and the other being the negative or interior one, so that at any point (x, y, z) of the surface, there exists two unit normal and opposite vectors n_1 and n_2, n_1 pointing towards the positive side, and n_2 towards the negative side; moreover n_1 and n_2 should be continuous. Thus to specify a side of $\mathcal{S}$, at any point of $\mathcal{S}$, we choose a unit normal vector $\vec{n}$ pointing towards the exterior side.*

Remarque 1.5.1 *This definition assumes that we can talk about the "two sides" of the surface $\mathcal{S}$.*

Let $\Phi : D \to \mathrm{I\!R}^3$ be a parametrization of an oriented surface $\mathcal{S}$.
We assume that $\mathcal{S}$ is regular at $\Phi(u_0, v_0)$, $(u_0, v_0) \in D$.
In this case, the vector $T_u \wedge T_v(u_0, v_0)$ is non zero, that is $\| T_u \wedge T_v(u_0, v_0) \| \neq 0$, and we have seen that the vector $T_u \wedge T_v(u_0, v_0)$ was normal to $\mathcal{S}$ at point $\Phi(u_0, v_0)$.
Thus we obtain a unit normal vector if we consider the vector $\frac{T_u \wedge T_v(u_0,v_0)}{\|T_u \wedge T_v(u_0,v_0)\|}$.
Since the surface $\mathcal{S}$ is oriented, we have made the choice of a normal vector field $\vec{n}$ always pointing in the same positive side. Finally, we must have

$$\frac{T_u \wedge T_v(u_0, v_0)}{\| T_u \wedge T_v(u_0, v_0) \|} = \mp\vec{n}(\Phi(u_0, v_0))$$

We are led to set

Définition 1.5.3 *With the above notations, we say that Φ preserves the orientation of $\mathcal{S}$, if we have always the sign $+$ in the above equality; that is the vector $T_u \wedge T_v$ always points towards the exterior (choosen once we are talking of oriented surface).*
On the other hand, if $T_u \wedge T_v$ points always towards the interior of $\mathcal{S}$, we say that Φ reverses the orientation, that is we have always the negative sign $-$ in the above equality.

Exemple 1.5.2 *Consider the unit sphere $\mathcal{S} : x^2 + y^2 + z^2 = 1$. We choose the exterior side of $\mathcal{S}$ as the positive side. This amounts to say that we have choosen as unit normal vector field the vector $\vec{r} = (x, y, z)$ which always points towards the exterior of $\mathcal{S}$. Let Φ be the parametrization of $\mathcal{S}$ given by $D = \{0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}$ and*

$$x = \cos\theta \sin\phi, y = \sin\theta \sin\phi, z = \cos\phi$$

Computing, we find that $T_\theta \wedge T_\phi = -r \sin\phi$. As $\sin\phi \leq 0$, this means that $T_\theta \wedge T_\phi$ points always towards the interior. Thus Φ reverses the orientation.

Exemple 1.5.3 *Let $\mathcal{S}$ be the graph of a function g. We have seen that a normal vector at point (x,y,z) of $\mathcal{S}$ was given by*

$$T_u \wedge T_v = (-\partial_u g, -\partial_v g, 1)$$

Thus we obtain two unit normal vectors by setting

$$\vec{n} = \mp[(\partial_u g)^2 + (\partial_v g)^2 + 1]^{\frac{1}{2}}(-\partial_u g, -\partial_v g, 1)$$

The third component is always positive (if we choose the sign +). Finally, we may always choose an orientation for $\mathcal{S}$ by choosing as + side, the side where points vector $\vec{n}$.
If we make this choice, then Φ preserves the orientation.

The interest in this notion of orientation lies in the following result:

Théorème 1.5.1 *Let $\mathcal{S}$ be an oriented surface, and F a continuous vector field defined over $\mathcal{S}$.*
1) If Φ_1 and Φ_2 are two (injective) parametrizations preserving the orientation of $\mathcal{S}$, then

$$\int\int_{\Phi_1} F.dS = \int\int_{\Phi_2} F.dS$$

2) If Φ_1 and Φ_2 are two (injective) parametrizations reversing the orientation of $\mathcal{S}$, then

$$\int\int_{\Phi_1} F.dS = -\int\int_{\Phi_2} F.dS$$

Note carefully that in the case of scalar functions, we have always, if $f : \mathbb{R}^3 \to \mathbb{R}$

$$\int\int_{\Phi_1} f dS = \int\int_{\Phi_2} f dS$$

We can thus introduce

Définition 1.5.4 *1) Case of surface integrals of scalar functions: let $\mathcal{S}$ be a parametrized surface. Let $f : I\!R^3 \to I\!R$ be a continuous function. Then, by definition, we set*

$$\int\int_{\mathcal{S}} f dS = \int\int_{\Phi} f dS$$

where Φ is any parametrization of $\mathcal{S}$, but satisfying assumptions (1.3.2).

2) Case of integrals of vectorial functions: let $\mathcal{S}$ be an oriented parametrized surface. Let $F : I\!R^3 \to I\!R^3$ be a vector field of class C^0. Then we set

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\Phi} F.dS$$

where Φ is any parametrization of $\mathcal{S}$, satisfying assumptions (1.3.2) and preserving the orientation of $\mathcal{S}$. Similarly, we set

$$\int\int_{\mathcal{S}^-} F.dS = \int\int_{\Phi} F.dS$$

where Φ is any parametrization of $\mathcal{S}$, satisfying assumptions (1.3.2) and reversing the orientation of $\mathcal{S}$.

In conclusion, we shall denote, with the above notations

$$\int\int_{\mathcal{S}^+} F.dS = -\int\int_{\mathcal{S}^-} F.dS$$

The number $\int\int_{\mathcal{S}^+} F.dS$ is also called the flux of F across the positively oriented surface $\mathcal{S}$. To end up, here is a small link between surface integral of scalar and vectorial functions. Consider $\mathcal{S}$ an oriented and regular surface, with Φ a parametrization preserving the orientation. Thus $n = \frac{T_u \wedge T_v}{\|T_u \wedge T_v\|}$ is the unit normal vector pointing towards the exterior of $\mathcal{S}$ (this is the positive side). We deduce that

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\Phi} F.dS = \int\int_{D} F.(T_u \wedge T_v) dudv =$$

$$= \int\int_{D} (\frac{T_u \wedge T_v}{\| T_u \wedge T_v \|}) \| T_u \wedge T_v \| dudv =$$

$$= \int\int_{D} (F.n) \| T_u \wedge T_v \| dudv = \int\int_{\mathcal{S}} (F.n) dS$$

Thus we have

Proposition 1.5.1 *With the above notations, we have*

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_{\mathcal{S}} (F.n) dS$$

Careful: the first integral is a flux, that is the integral of a vectorial function, here F, while the second integral is a surface integral of a scalar function, here $F.n$.
If we apply this to the case of a surface $\mathcal{S}$ given by the graph of a function g, we obtain

$$\int\int_{\mathcal{S}^+} F.dS = \int\int_D [F_1(-\partial_u g) + F_2(-\partial_v g) + F_3] du dv$$

where $F = (F_1, F_2, F_3)$ are the components of the vector field.

1.6 Exercices of this Chapter

1. Find a cartesian equation for the tangent plane to each of the following surfaces at the given points $(u, v) \in \mathbb{R}^2$:
   (a) $x = 2u, y = u^2 + v, z = v^2$ at $(0, 1, 1)$.
   (b) $x = u^2 - v^2, y = u + v, z = u^2 + 4v$ at $(\frac{-1}{4}, \frac{1}{2}, 2)$.
   (c) $x = u^2, y = u \sin e^v, z = \frac{1}{3} u \cos e^v$ at $(13, -2, 1)$.
2. Are the surfaces of exercice 1 a) and b) regular?
3. Find an expression of a unit normal vector to each of the following surfaces:
   (a) $c = \cos v \sin u, y = \sin v \sin u, z = \cos u, (u, v) \in [0, \pi] \times [0, 2\pi]$.
   (b) $x = \sin v, y = u, z = \cos v, u \in [-1, 3], v \in [0, 2\pi]$
4. Find the area of the surface S in each case:
   (a) S is the unit sphere paramtrized by $\Phi : D \to S \subset \mathbb{R}^3$, where D is the rectangle $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi$ and Φ given by

   $$x = \cos\theta \sin\phi, y = \sin\theta \sin\phi, z = \cos\phi$$

   (b) idem but $0 \leq \phi \leq 2\pi$.
   (c) idem but $-\pi/2 \leq \phi \leq \pi/2$.
   (d) S is the torus, that is D is the rectangle $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq 2\pi$ and

   $$x = (1 + \cos\phi)\cos\theta, y = (1 + \cos\phi)\sin\theta, z = \sin\phi$$

   (e) D is the unit disk of R^2 and

   $$x = u - v, y = u + v, z = uv$$

5. Surface integrals of scalar functions. Compute:
   (a) $\int\int_S zdS$ where S is the upper half sphere of radius 2.
   (b) $\int\int_S (x+y+z)dS$ where S is the unit sphere.
   (c) $\int\int_S zdS$ where S is the surface $z = x^2+y^2$, $x^2+y^2 \leq 1$.

6. Let$S$ the sphere of radius r and P a point out of S. Denote by B the unit ball. Show that
$$\int\int_S \frac{1}{\parallel X-P \parallel} dS = \begin{cases} 4\pi r \ if \ P \in B, \\ 4\pi r^2/d \ if \ P \notin B, \end{cases}$$
where d is the distance from P to the origin.

7. Compute $\int\int_S rotF.dS$, where $F = (y, -x, zx^3y^2)$ and S is the surface $x^2+y^2+3z^2 = 1$, $z \leq 0$. (n is the unit normal vector pointing upwards).

8. Compute $\int\int_S rotF.dS$, where $F = (x^2+y-4, 3xy, 2xz+z^2)$ and S is the surface $x^2+y^2+z^2 = 16$, $z \geq 0$. (n is the unit normal vector pointing upwards).

9. Compute $\int\int_S F.dS$, where $F = (3xy^2, 3x^2y, z^3)$ and S is the unit sphere.

10. Let S be the unit sphere. Let F be a vector field and F_r its radial component. Show that
$$\int\int_S F.dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} F_r \sin\phi d\phi d\theta.$$

11. Let $a > 0$, $b > 0$ and $c > 0$. Consider the following subset
$$S = \{(x,y,z), \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1, z \geq 0\}$$
oriented with the normal pointing upwards. Compute $\int\int_S F.dS$ where $F = (x^3, 0, 0)$.

12. Let S be the half sphere $\{(x,y,z), x^2+y^2+z^2 = 1, z \geq 0\}$ oriented with the exterior normal, Compute $\int\int_S F.dS$ in the following cases:
    (a) $F = (x, y, 0)$
    (b) $F = (y, x, 0)$
    (c) For these two cases, compute $\int\int_S (rotF).dS$ and $\int_C F.ds$ where C is the unit circle in the plane xy, described counterclockwise (seen from the positive z axis). Note that C is the boundary of S.

13. (a) Let $F = gradf$, for a given scalar function f . Let c be a closed path. Show that $\int_c F.ds = 0$.

(b) Let S be a surface with frontier c. Show that

$$\int\int_S (rot F).dS = \int_c F.ds$$

if F is a gradient vector field.